A Markovian characterization of the exponential twist of probability measures

Thibaut Bourdais[1,a], Nadia Oudjane[2,b] and Francesco Russo [3,c]

[1] *Unité de Mathématiques appliquées, ENSTA Paris, Institut Polytechnique de Paris, Palaiseau, France,* [a]*thibaut.bourdais@ensta-paris.fr*
[2] *EDF R&D and FiME (Laboratoire de Finance des Marchés de l'Energie (Dauphine, CREST, EDF R&D) www.fime-lab.org),* [b]*nadia.oudjane@edf.fr*
[3] *Unité de Mathématiques appliquées, ENSTA Paris, Institut Polytechnique de Paris, Palaiseau, France,* [c]*francesco.russo@ensta-paris.fr*

Abstract. In this paper we study the exponential twist, i.e. a path-integral exponential change of measure, of a Markovian reference probability measure $\mathbb{P}$. This type of transformation naturally appears in variational representation formulae originating from the theory of large deviations and can be interpreted in some cases, as the solution of a specific stochastic control problem. Under a very general Markovian assumption on $\mathbb{P}$, we fully characterize the exponential twist probability measure as the solution of a martingale problem and prove that it inherits the Markov property of the reference measure. The "generator" of the martingale problem shows a drift depending on a *generalized gradient* of some suitable *value function* v.

Résumé. Dans ce papier on étudie l'*exponential twist*, c'est-à-dire un changement de probabilité exponentiel exprimé par une intégrale de chemin, d'une mesure de probabilité markovienne de référence $\mathbb{P}$. Ce type de transformation apparaît dans des formules de représentation variationnelles issues de la théorie des grandes déviations, et peut s'interpréter dans certains cas comme la solution d'un problème de contrôle stochastique. Sous une hypothèse markovienne très générale sur $\mathbb{P}$, on caractérise complètement l'*exponential twist* de $\mathbb{P}$ comme la solution d'un problème de martingale et on montre que cette nouvelle mesure hérite de la propriété de Markov de la mesure de référence. Le générateur de ce problème de martingale fait apparaître un drift qui s'exprime comme un *gradient généralisé* d'une *fonction valeur v* bien choisie.

MSC2020 subject classifications: Primary 60H30, 60J60, 65C05; secondary 49L25, 35K58
Keywords: Stochastic control, optimization, exponential twist, generalized gradient, relative entropy

1. Introduction

This paper focuses on exponential twist probability measures $\mathbb{Q}^*$ resulting from an exponential change of measure with respect to a Markovian reference probability measure $\mathbb{P}$, i.e.

$$d\mathbb{Q}^* \propto e^{-\varphi} d\mathbb{P}, \tag{1.1}$$

when φ is a *path-integral* functional of the form

$$\varphi(X) = \int_0^T f(r, X_r)dr + g(X_T), \tag{1.2}$$

for measurable functions f, g. More precisely we will assume that the reference probability measure $\mathbb{P}$ is a solution of a càdlàg Markovian martingale problem, including the case when $\mathbb{P}$ is the law of a stochastic differential equation (SDE) with jumps and possibly singular coefficients: the drift could even be distributional. The notion of martingale problem was introduced by D.W. Stroock and S.R.S Varadhan in the seminal papers [41, 42] and has been exploited extensively starting from [43].

The objective of the paper is then to completely characterize $\mathbb{Q}^*$ as a solution of a martingale problem, which is still Markovian, under very general assumptions on $\mathbb{P}$. The properties of $\mathbb{Q}^*$ have been extensively studied in [8] when the reference probability measure $\mathbb{P}$ is the Wiener measure. In the case of discrete time Markov models with finite state space, the stability of the Markov property by the exponential twist transformation (1.1) was already pointed out in [12, 13, 31]. In this paper we extend these results to càdlàg (possibly singular) Markovian models and provide a precise characterization of the generator associated to the martingale problem verified by the probability measure $\mathbb{Q}^*$.

Exponential twist probability measures of the form (1.1) are intimately connected to various application domains. It appears naturally in variational representation formulae in relation to the theory of large deviations [19, 37]. In fact we have the variational formula

$$-\log \int_\Omega e^{-\varphi(\omega)} d\mathbb{P}(\omega) = \inf_{\mathbb{Q}\in\mathcal{P}(\Omega)} \int_\Omega \varphi(\omega) d\mathbb{Q}(\omega) + H(\mathbb{Q}|\mathbb{P}), \tag{1.3}$$

see Proposition 1.4.2 in [19], where $\mathcal{P}(\Omega)$ is the space of all probability measures on $(\Omega, \mathcal{F})$ and H is the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$, see Definition 2.4. The minimum in (1.3) is achieved for the exponential twist probability measure $d\mathbb{Q}^* \propto e^{-\varphi} d\mathbb{P}$ and $\mathbb{Q}^*$ is said to be a *solution* to the optimization problem (1.3). Furthermore, the minimization Problem (1.3) can often be reinterpreted as a stochastic optimal control problem. For instance, when $\mathbb{P}$ is the law of an SDE of the type

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r, \quad \forall t \in [0, T]$$

where T is a fixed time horizon, (1.3) is equivalent to

$$\begin{cases} \inf_{u\in\mathcal{U}} \mathbb{E}\left[\int_0^T f(r, X_r^{u,x})dr + \frac{1}{2}\int_0^T |u_r|^2 dr + g(X_T^{u,x})\right] \\ dX_r^{u,x} = b(r, X_r^{u,x})dr + \sigma(r, X_r^{u,x})u_r dr + \sigma(r, X_r^{u,x})dW_r \\ X_0^{u,x} = x, \end{cases} \tag{1.4}$$

where $\mathcal{U}$ is a set of progressively measurable processes u such that the controlled SDE $dX_r^{u,x} = b(r, X_r^{u,x})dr + \sigma(r, X_r^{u,x})u_r dr + \sigma(r, X_r^{u,x})dW_r,\ X_t^{u,x} = x$ has a solution, see e.g. [9]. This equivalence between Problems (1.3) and (1.4), first stated in [21, 22], has, since then, given rise to several developments not only in the field of large deviations but also in concrete applications, where it is used to derive efficient methods to approximate the solution of the optimal control Problem (1.4), see [12, 44, 45]. The specific setting (1.4) is often referred to as *Path integral control* in this applications literature. Moreover, the characterization results for the exponential twist measure (1.1) provided in the present paper apply to the framework of non-linear optimization on the space of probability measures (often related to mean-field optimization [14]) stated as

$$\inf_{\mathbb{Q}\in\mathcal{P}(\Omega)} F\left(\mathbb{E}^{\mathbb{Q}}[\varphi(X)]\right) + H(\mathbb{Q}|\mathbb{P}), \tag{1.5}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a differentiable convex function and φ is given by (1.2). Indeed, assume that Problem (1.5) admits a solution $\mathbb{Q}^*$. Then $\mathbb{Q}^*$ is also solution of

$$\inf_{\mathbb{Q}\in\mathcal{P}(\Omega)} \mathbb{E}^{\mathbb{Q}}[\tilde{\varphi}(X)] + H(\mathbb{Q}|\mathbb{P}), \tag{1.6}$$

where $\tilde{\varphi}(X) := F'\left(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right)\varphi(X)$ (see Lemma E.1) and $\mathbb{Q}^*$ is an exponential twist measure of the form (1.1) with $\varphi = \tilde{\varphi}$. Hence, any solution of Problem (1.5) falls into the framework of the present paper. Optimization programs of the form (1.5) appear for example in [12, 13, 38, 39] for demand side management in power systems.

The main result of the paper is stated in Section 3, where we identify a significant Ideal Condition 3.12 associated with the reference probability measure $\mathbb{P}$ and a functional domain $\mathcal{D}$ on which we characterize the generator $a^{\mathbb{Q}^*}$ of the martingale problem verified by $\mathbb{Q}^*$. More precisely, under the Ideal Condition 3.12 the exponential twist probability measure $\mathbb{Q}^*$ given by (1.1) is characterized as a solution of a well-identified martingale problem in Theorem 3.15. For all test function $\phi \in \mathcal{D}$, $a^{\mathbb{Q}^*}(\phi)$ can be written as $a^{\mathbb{P}}(\phi) + \Gamma^v(\phi)/v$ where $a^{\mathbb{P}}$ is the generator associated to the martingale problem verified by $\mathbb{P}$ and $\Gamma^v(\phi)/v$ is a correction term identified via a Girsanov's change of measure.

In Section 4, we further specify the map $\mathcal{D} \ni \phi \mapsto \Gamma^v(\phi)$ in the integro-differential case. In particular Proposition 4.4 states that it can be expressed as the sum of an integral term corresponding to the jumps contribution, and a *generalized gradient* of v. Then, in Proposition 4.11, we extend the continuous part $\Gamma^{v,c}$ of this operator to a larger space $\mathbb{D}$ including the identity function id and other test functions outside the domain $\mathcal{D}$. This extension allows us, in Proposition 4.12, to express the change of probability measure as a drift modification depending only on the continuous part of the (extended) operator evaluated in the identity function i.e. $\Gamma^{v,c}(id)$. In particular, even when the initial drift is a Schwartz distribution, an additional measurable drift term appears, extending the term $\sigma\sigma^\top \nabla_x v$ when $\nabla_x v$ does not exist.

In Section 5 we provide general conditions under which the Ideal Condition 3.12 holds. We require our reference probability measure $\mathbb{P}$ to fulfill a slightly stronger version of the Markov property, namely to be *Regularly Markovian* in the sense of Definition 5.7. This regularity assumption on $\mathbb{P}$ allows to build an extension of the Carré du champ operator, see Propositions 5.11, in order to identify the generator of the martingale problem satisfied by $\mathbb{Q}^*$. This extension was initiated in [4] in the case of genuine martingales and used to define Pseudo-Partial Differential Equations (Pseudo-PDE) and their associated probabilistic representations via martingale driven backward SDEs in [5–7]. In Appendix A we provide an alternative method, in the case of Brownian diffusions, for verifying the Ideal Condition, based on the existence of a solution of one PDE.

In Section 6, we instantiate our characterization result (Theorem 3.15) on several specific examples using the sufficient assumptions provided in Section 5. We first study the case where the reference probability measure $\mathbb{P}$ is solution to a martingale problem associated to a jump diffusion. In this situation we are able to fully characterize in Proposition 6.4 the drift of the canonical process under $\mathbb{Q}^*$, as well as its jump intensity, as Markovian functions of the current state. We emphasize once again that we do not require any integrability condition of the underlying process with respect to $\mathbb{P}$. We then apply these results to the case of Brownian diffusions which corresponds to the stochastic control problem (1.4), as already stated. Relying on this equivalence and on our characterization Theorem 3.15, we derive in Corollary 6.7 the existence of an optimal Markovian control for Problem (1.4) in a very general framework, and we characterize this optimal control by means of a generalized gradient. In comparison with compactification methods used e.g. in [20, 23, 24], we do not require any semi-continuity hypotheses for the cost functions f and g (only measurability in the space variable), nor any integrability condition on the controlled process to prove the existence of an optimal Markovian controls. We finally consider more irregular examples in Proposition 6.13 where the drift b is a distribution.

In the companion paper we make use of Corollary 6.11 which is a consequence of Corollary 6.7 and Lemma 6.10. This is a basic tool which allows us to develop an algorithm that provides Markovian controls approximating the solution to a large class of stochastic control problems.

Once we completed this paper, we discovered the article [34] which provides a similar statement to the one of Theorem 3.15. In particular the authors formulate in their Theorem 4.2 a similar hypothesis to our Ideal Condition, assuming the existence of a so called *good function*. A comparison between the two results is discussed in Remark 3.17. However, we recall that we provide two additional major contributions in our paper. In Section 4, we further characterize the exponential twist change of measure by identifying the change of drift via a generalized gradient, while in Section 5 we provide a general framework in which the key Ideal Condition is verified, based on a so-called regular Markovian assumption.

2. Notations and definitions

In this section we introduce the basic notions and notations used throughout this document. In what follows, $T \in \mathbb{R}^+$ will be a fixed time horizon.

- All vectors $x \in \mathbb{R}^d$ are column vectors. Given $x \in \mathbb{R}^d$, $|x|$ will denote its Euclidean norm.
- Given a matrix $A \in \mathbb{R}^{d\times d}$, $\|A\| := \sqrt{Tr[AA^\top]}$ will denote its Frobenius norm.
- For any $x \in \mathbb{R}^d$, δ_x will denote the Dirac mass in x.
- $\mathcal{C}^{i,j} := C^{i,j}([0,T]\times\mathbb{R}^d,\mathbb{R})$ will be the space of real valued functions on $[0,T]\times\mathbb{R}^d$ that are continuous together with their time and space derivatives up to order i and j respectively. It is endowed with the topology of uniform convergence on compact sets (u.c.s).
- $\mathcal{C}^{i,j}_b := C^{i,j}_b([0,T]\times\mathbb{R}^d,\mathbb{R})$ will be the space of real valued functions on $[0,T]\times\mathbb{R}^d$ that are bounded and continuous together with their time and space derivatives up to order i and j respectively. It is endowed with the topology of uniform convergence.
- For any topological spaces E and F, $\mathcal{B}(E)$ will denote the Borel σ-field of E. $C(E,F)$ (resp. $C_b(E,F)$, $\mathcal{B}(E,F)$, $\mathcal{B}_b(E,F)$ will denote the linear space of functions from E to F that are continuous (resp. bounded continuous, Borel, Borel bounded, Borel locally bounded). If $E = F$ we will simply denote $C(E)$ (resp. $C_b(E)$, $\mathcal{B}(E)$) for $C(E,E)$ (resp. $C_b(E,E)$, $\mathcal{B}(E,E)$). $\mathcal{P}(E)$ will denote the set of Borel probability measures on E. Given $\mathbb{P} \in \mathcal{P}(E)$, $\mathbb{E}^{\mathbb{P}}$ will denote the expectation with respect to (w.r.t.) $\mathbb{P}$.
- Given $\phi \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, $\partial_t\phi$, $\nabla_x\phi$ and $\nabla^2_x\phi$ will denote respectively the partial derivative of ϕ with respect to $t\in[0,T]$, its gradient and its Hessian matrix w.r.t. $x\in\mathbb{R}^d$. Given any bounded function ϕ we will denote by $|\phi|_\infty$ its supremum.
- For $x \in \mathbb{R}^d$, $id(x) := (id_i(x))_{1\le i\le d} := (x_i)_{1\le i\le d}$ will denote the identity on $\mathbb{R}^d$.
- Given $0\le t\le T$, $D([t,T],\mathbb{R}^d)$ will denote of càdlàg functions defined on $[t,T]$ with values in $\mathbb{R}^d$. In the whole paper Ω will denote space $D([0,T])$. For any $t\in[0,T]$ we denote by $X_t : \omega\in\Omega\mapsto\omega_t$ the coordinate mapping on Ω. We introduce the σ-field $\mathcal{F} := \sigma(X_r, 0\le r\le T)$. On the measurable space $(\Omega,\mathcal{F})$, we introduce the *canonical process* $X : \omega\in([0,T]\times\Omega,\mathcal{B}([0,T])\otimes\mathcal{F})\mapsto X_t(\omega)=\omega_t\in(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$.
  We endow $(\Omega,\mathcal{F})$ with the right-continuous filtration $\mathcal{F}_t := \bigcap_{t<s\le T}\sigma(X_r, 0\le r\le s)$. The filtered space $(\Omega,\mathcal{F},(\mathcal{F}_t))$ will be called the *canonical space* (for the sake of brevity, we denote $(\mathcal{F}_t)_{t\in[0,T]}$ by $(\mathcal{F}_t)$).
  For $0\le t\le u\le T$, we denote $\mathcal{F}_{t,u} := \bigcap_{u<v\le T}\sigma(X_r, u\le r\le v)$.
- Given $\mathbb{P}\in\mathcal{P}(\Omega)$ and a generic σ-field $\mathcal{G}$ on Ω, $\mathcal{G}^{\mathbb{P}}$ will denote the $\mathbb{P}$-completion of $\mathcal{G}$.
- Given $\mathbb{P}\in\mathcal{P}(\Omega)$, $\mathbb{D}^{ucp}(\mathbb{P})$ will denote the space of all càdlàg adapted processes (indexed by $[0,T]$) endowed with topology of the uniform convergence in probability (u.c.p.) topology under $\mathbb{P}$.
- Given $\mathbb{P}\in\mathcal{P}(\Omega)$, $\mathcal{H}^2_{loc}(\mathbb{P})$ will denote the space of locally square-integrable martingales. Given $M,N\in\mathcal{H}^2_{loc}(\mathbb{P})$, $\langle M,N\rangle$ will denote their predictable *angle bracket*. If $M=N$, we will use the notation $\langle M\rangle$. We also denote $Pos(\langle M,N\rangle) := \frac{1}{4}\langle M+N\rangle$ and $Neg(\langle M,N\rangle) := \frac{1}{4}\langle M-N\rangle$.
- Given $\mathbb{P}\in\mathcal{P}(\Omega)$, $\mathcal{A}_{loc}(\mathbb{P})$ will denote the set of càdlàg processes with $\mathbb{P}$-locally integrable variation.
- Equality between stochastic processes are in the sense of *indistinguishability*.
- Throughout the paper we will use the notion of random measures and their associated compensator. For a detailed discussion on this topic as well as some unexplained notations we refer to Chapter II and Chapter III in [26].
- We will work with the convention that $\inf\emptyset = +\infty$. In particular, any hitting time τ of a Borel set by a stochastic process defined on $[0,T]$ will have values in $[0,T]\cup\{+\infty\}$.

Definition 2.1. *(Lévy kernel). $L : [0,T]\times\Omega\times\mathcal{B}(\mathbb{R}^d)$ is called a Lévy kernel if it satisfies the following.*

1. *$L(t,X,.)$ is a non-negative Borel measure on $\mathbb{R}^d$ such that $L(t,X,\{0\})=0$, which is σ-finite on $\mathbb{R}^d\backslash\{0\}$.*

2. $(t, X) \mapsto L(t, X, A)$ *is predictable for all* $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$*;*
3. $(t, X) \mapsto \int_A (1 \wedge |q|^2) L(t, X, dq)$ *is Borel and bounded for all* $A \in \mathcal{B}(\mathbb{R}^d)$.

Definition 2.2. *Let* $\mathbb{P} \in \mathcal{P}(\Omega)$*. Let* Y, Z *be a càdlàg process.*

1. *(Covariation). We define*

$$[Z, Y]^\varepsilon(t) := \int_{]0,\, t]} \frac{(Z((r+\varepsilon) \wedge t) - Z(r))(Y((r+\varepsilon) \wedge t) - Y(r))}{\varepsilon}\, dr. \tag{2.1}$$

   $[Z, Y]$ *is by definition the u.c.p. limit, whenever it exists, of* $[Z, Y]^\varepsilon$ *when* $\epsilon \to 0$*. If* Y, Z *are càdlàg semimartingales then* $[Y, Z]$ *is the usual (quadratic) covariation, see Proposition 1.1 of [35].*
2. *(Weak Dirichlet process).* Z *is called a weak Dirichlet process if it is* $(\mathcal{F}_t)$*-adapted and if under* $\mathbb{P}$ *it admits a decomposition* $Z = M + A$ *where* M *is a* $(\mathbb{P}, \mathcal{F}_t)$*-local martingale and the process* A *satisfies* $[A, N] = 0$ *for all* $(\mathbb{P}, \mathcal{F}_t)$*-continuous local martingale.* A *will be called a martingale orthogonal process. For more properties on those processes, see [36], Chapter 15 and [2, 3]. In particular an* $(\mathcal{F}_t)$*-semimartingale is a weak Dirichlet process.*
3. *A multidimensional weak Dirichlet process is a multidimensional process such that every component is a weak Dirichlet process.*

The following statement is Proposition 3.2 in [3].

Proposition 2.3. *Let* Z *be a càdlàg weak Dirichlet process. There exists a unique continuous local martingale* Z^c *and a unique process* A*, vanishing at zero, verifying* $[A, N] = 0$ *for all* $(\mathbb{P}, \mathcal{F}_t)$*-continuous local martingale such that* $Z = Z^c + A$*.*

Definition 2.4. *(Relative entropy). Let* E *be a topological space. Let* $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(E)$*. The relative entropy* $H(\mathbb{Q}|\mathbb{P})$ *between the measures* $\mathbb{P}$ *and* $\mathbb{Q}$ *is defined by*

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}\left[\log \frac{d\mathbb{Q}}{d\mathbb{P}}\right] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

with the convention $\log(0/0) = 0$.

Remark 2.5. *The relative entropy* H *fulfills the following properties for which we refer to [19] Lemma 1.4.3.*

1. H *is* ***non negative*** *and* ***jointly convex****, that is for all* $\mathbb{P}_1, \mathbb{P}_2, \mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{P}(E)$*, for all* $\lambda \in [0, 1]$*,* $H(\lambda\mathbb{Q}_1 + (1-\lambda)\mathbb{Q}_2 | \lambda\mathbb{P}_1 + (1-\lambda)\mathbb{P}_2) \le \lambda H(\mathbb{Q}_1|\mathbb{P}_1) + (1-\lambda) H(\mathbb{Q}_2|\mathbb{P}_2)$.
2. $(\mathbb{P}, \mathbb{Q}) \mapsto H(\mathbb{Q}|\mathbb{P})$ *is lower semicontinous with respect to the weak convergence on Polish spaces.*

We will often use the notion of martingale problem.

Definition 2.6. *(Martingale problem). Let* $a : \mathcal{D} \subset \mathcal{B}([0, T] \times \mathbb{R}^d, \mathbb{R}) \to \mathcal{B}([0, T] \times \mathbb{R}^d, \mathbb{R})$ *be a linear operator. Let* $\mu \in \mathcal{P}(\mathbb{R}^d)$*. We say that a probability measure* $\mathbb{P} \in \mathcal{P}(\Omega)$ *is solution of the martingale problem associated to* $(\mathcal{D}, a, \mu)$ *if*

1. $\mathcal{L}^{\mathbb{P}}(X_0) = \mu$*;*
2. *for every* $\phi \in \mathcal{D}$ *the process*

$$M[\phi] := \phi(\cdot, X_\cdot) - \phi(0, X_0) - \int_0^\cdot a(\phi)(r, X_r) dr \tag{2.3}$$

   has a càdlàg modification which is a local martingale under $\mathbb{P}$.

3. Characterization of the exponential twist measure

We consider f, g verifying Hypothesis 3.1 below.

Hypothesis 3.1. *(Cost functions).* $f \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R})$, $g \in \mathcal{B}(\mathbb{R}^d, \mathbb{R})$ *and* $f, g \geq 0$.

Remark 3.2. *Without restriction of generality, Hypothesis 3.1 can be relaxed supposing* f, g *to be lower bounded.*

We will moreover assume that the reference probability measure $\mathbb{P}$ has the Markov property (3.3) below. We introduce to this aim the following assumption.

Hypothesis 3.3. $\mathbb{P}$ *satisfies the Markov property*

$$\mathbb{E}^{\mathbb{P}}[F((X_u)_{u\in[t,T]})|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[F((X_u)_{u\in[t,T]})|X_t], \tag{3.1}$$

for all $F \in \mathcal{B}_b(D[t,T], \mathbb{R})$, $t \in [0,T]$.

In the rest of the paper, we assume that Hypotheses 3.1 and 3.3 are verified.

In relation to previous Hypothesis 3.3 we introduce the notion of Markov domain below, defined for a probability measure $\mathbb{P}$.

Definition 3.4. *(Markov domain* $\mathcal{D}(\mathbb{P})$*). A Borel locally bounded function* $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is an element of the Markov domain* $\mathcal{D}(\mathbb{P})$ *if there exists a function* $\chi \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R})$ *such that the process*

$$M[\phi] := \phi(\cdot, X_\cdot) - \phi(0, X_0) - \int_0^{\cdot} \chi(r, X_r)dr \tag{3.2}$$

has a càdlàg modification in $\mathcal{H}^2_{loc}(\mathbb{P})$*. We will denote* $a^{\mathbb{P}}(\phi) := \chi$*. That modification will still be denoted* $M[\phi]$*.*

We introduce here a significant space of (equivalence classes) of Borel functions.

Notation 3.5.

$$L^0 := L^0(\mathbb{P}) = \left\{ \phi \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R}) \ : \ \int_0^T |\phi(r, X_r)|dr < +\infty \quad \mathbb{P}\text{-a.s.} \right\}, \tag{3.3}$$

which corresponds to the classical space $L^0([0,T] \times \mathbb{R}^d, dt \otimes d\mathbb{P}_{X_t})$*, where* P_{X_t} *is the (marginal) law of* X_t *under* $\mathbb{P}$*. With a slight abuse of notations* L^0 *can be seen as a linear space of equivalence classes, where the equivalence is given by the equality up to a* $dt \otimes d\mathbb{P}_{X_t}$ *null set.*

Remark 3.6.
1. $a^{\mathbb{P}}(\phi)$ *defines a* $dt \otimes d\mathbb{P}_{X_t}$*-unique element of* L^0*. Indeed assume that there exist two elements* χ_1 *and* χ_2 *of* $\mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R})$ *such that* (3.2) *holds for* $\chi = \chi_1$ *or* χ_2*. Clearly* $\phi(\cdot, X_\cdot)$ *is a special semimartingale under* $\mathbb{P}$ *and uniqueness of the decomposition of special semimartingales immediately yields* $\int_0^t \chi_1(r, X_r)dr = \int_0^t \chi_2(r, X_r)dr$ $\mathbb{P}$*-a.s. for all* $t \in [0,T]$*, that is* $\chi_1 = \chi_2$ $dt \otimes d\mathbb{P}_{X_t}$*-a.e.*
2. *Also if* $\phi^1 = \phi^2$ *in* L^0 *then* $M[\phi^1] = M[\phi^2]$ *up to a modification.*
3. *Assume that* $\mathbb{P}$ *is solution of a martingale problem associated to* $(\mathcal{D}, a)$ *in the sense of Definition 2.6. If* $\mathbb{P}$ *fulfills the martingale problem with respect to* $(\mathcal{D}, a)$*, then* $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$ *and* a *is a restriction of* $a^{\mathbb{P}}$ *to* $\mathcal{D}$*.*
4. *If* $\lambda, \mu \in \mathbb{R}$ *and* $\phi^1, \phi^2 \in \mathcal{D}$ *then (up to a modification)* $M[\lambda\phi^1 + \mu\phi^2] = \lambda M[\phi^1] + \mu M[\phi^2]$*.*
5. *When* $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$ *is a linear subspace of* $\mathcal{D}(\mathbb{P})$ *which is also an algebra (i.e. stable by multiplication),* $\mathcal{D}$ *will be called a subalgebra of* $\mathcal{D}(\mathbb{P})$*.*

We will need later the following technical result.

Lemma 3.7. *Let* ψ *and* ϕ *be two elements of* $\mathcal{D}(\mathbb{P})$*. The following statements are equivalent.*

1. $\phi\psi \in \mathcal{D}(\mathbb{P})$*.*
2. *There exists* $\Gamma(\phi, \psi) \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R})$ *(unique in* L^0*) such that*

$$\langle M[\phi], M[\psi] \rangle = \int_0^{\cdot} \Gamma(\phi, \psi)(r, X_r)dr \quad \mathbb{P}\text{-a.s.}$$

Moreover we have

$$\Gamma(\phi,\psi) = a^{\mathbb{P}}(\phi\psi) - \psi a^{\mathbb{P}}(\phi) - \phi a^{\mathbb{P}}(\psi). \tag{3.4}$$

Remark 3.8. *The bilinear map* Γ *in* (3.4) *is called the Carré du champ operator.*

Proof of Lemma 3.7. Let $\phi,\psi \in \mathcal{D}(\mathbb{P})$. By integration by parts, for all $t \in [0,T]$,

$$\begin{aligned}\phi(t,X_t)\psi(t,X_t) &= \phi(0,X_0)\psi(0,X_0) + \int_0^t \phi(r,X_r)d\psi(r,X_r) + \int_0^t \psi(r,X_r)d\phi(r,X_r) + [M[\phi],M[\psi]]_r \\ &= \phi(0,X_0)\psi(0,X_0) + \int_0^t \phi(r,X_r)dM[\psi]_r + \int_0^t \phi(r,X_r)a^{\mathbb{P}}(\psi)(r,X_r)dr \\ &\quad + \int_0^t \psi(r,X_r)dM[\phi]_r + \int_0^t \psi(r,X_r)a^{\mathbb{P}}(\phi)(r,X_r)dr + [M[\phi],M[\psi]]_r.\end{aligned} \tag{3.5}$$

We now prove the first implication 1. $\Rightarrow$ 2. As $\phi\psi \in \mathcal{D}(\mathbb{P})$, we also have that

$$\phi(t,X_t)\psi(t,X_t) = \phi(0,X_0)\psi(0,X_0) + M[\phi\psi] + \int_0^t a^{\mathbb{P}}(\phi\psi)(r,X_r)dr. \tag{3.6}$$

Consequently, defining $\Gamma(\phi,\psi)$ as (3.4), we get

$$[M[\phi],M[\psi]] - \int_0^{\cdot} \Gamma(\phi,\psi)(r,X_r)dr = N, \tag{3.7}$$

where N is the local martingale

$$N := M[\phi\psi] - \int_0^{\cdot} \psi(r,X_r)dM[\phi]_r - \int_0^{\cdot} \phi(r,X_r)dM[\psi]_r.$$

By (3.7) and the fact that the process $\int_0^{\cdot} \Gamma(\phi,\psi)(r,X_r)dr$ is adapted continuous and therefore predictable, we conclude that $\langle M[\phi],M[\psi]\rangle = \int_0^{\cdot} \Gamma(\phi,\psi)(r,X_r)dr$ $\mathbb{P}$-a.s.

We now prove the converse implication 2. $\Rightarrow$ 1. By definition, the process

$$N^1 := [M[\phi],M[\psi]] - \langle M[\psi],M[\phi]\rangle = [M[\phi],M[\psi]] - \int_0^{\cdot} \Gamma(\phi,\psi)(r,X_r)dr,$$

is a local martingale. We deduce from (3.5) that

$$\phi(\cdot,X_\cdot)\psi(\cdot,X_\cdot) - \phi(0,X_0)\psi(0,X_0) - \int_0^{\cdot} \left(\Gamma(\phi,\psi) + \psi a^{\mathbb{P}}(\phi) + \phi a^{\mathbb{P}}(\psi)\right)(r,X_r)dr = M,$$

where

$$M := \int_0^{\cdot} \psi(r,X_r)dM[\phi]_r + \int_0^{\cdot} \phi(r,X_r)dM[\psi]_r + N^1,$$

is a $\mathbb{P}$-local martingale. At this point, it is clear that $\psi\phi \in \mathcal{D}(\mathbb{P})$ which yields 1. Moreover we get $a^{\mathbb{P}}(\phi\psi) = \Gamma(\phi,\psi) + \psi a^{\mathbb{P}}(\phi) + \phi a^{\mathbb{P}}(\psi)$, which finally implies (3.4). □

As mentioned earlier, in this paper we aim at characterizing the exponential twist $\mathbb{Q}^*$ defined by

$$d\mathbb{Q}^* := D_T d\mathbb{P}, \tag{3.8}$$

where

$$D_T := \frac{\exp\left(-\int_0^T f(r,X_r)dr - g(X_T)\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_0^T f(r,X_r)dr - g(X_T)\right)\right]}. \tag{3.9}$$

A first observation concerns the fact that the exponential twist $\mathbb{Q}^*$ conserves the Markov property, i.e. $\mathbb{Q}^*$ still fulfills Hypothesis 3.3, see Lemma B.6 in the Appendix. Besides the Markov property, as we will illustrate later, it is interesting to know the dynamics of the canonical process X under $\mathbb{Q}^*$, e.g. which martingale problem is fulfilled by $\mathbb{Q}^*$.

Since $\mathbb{P}$ verifies Hypothesis 3.3 (Markov property), for all $t \in [0,T]$, we have

$$\mathbb{E}^{\mathbb{P}}[D_T|\mathcal{F}_t] = \frac{\exp\left(-\int_0^t f(r,X_r)dr\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_0^T f(r,X_r)dr - g(X_T)\right)\right]}\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\bigg|X_t\right]. \tag{3.10}$$

Remark 3.9. *By Proposition 5.1 in [11] there exists a Borel function $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ such that*

$$v(t,X_t) := \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\bigg|X_t\right] \quad dt\otimes d\mathbb{P}\text{-a.e.} \tag{3.11}$$

v is then, obviously, unique as an element of L^0.

In the sequel we will make use of the notation

$$U^s_\cdot := 1_{[s,T]}(\cdot)\int_0^\cdot f(r,X_r)dr, \tag{3.12}$$

where $s \in [0,T]$. Below we denote by $D := (D_t)_{t\in[0,T]}$ the càdlàg version of the martingale $(\mathbb{E}^{\mathbb{P}}[D_T|\mathcal{F}_t])$, see (3.10).

Proposition 3.10. *Let $\mathbb{P}$ be a probability measure fulfilling the Markov property, i.e. Hypothesis 3.3. Let $v \in \mathcal{B}([0,T]\times\mathbb{R}^d,\mathbb{R})$ verifying* (3.11). *We set*

$$V_t = \mathbb{E}^{\mathbb{P}}[v(0,X_0)]\exp\left(\int_0^t f(r,X_r)dr\right)D_t, \quad t\in[0,T], \tag{3.13}$$

We have the following.

1. *V is a càdlàg version of $(v(t,X_t))$, i.e.*

$$V_t = v(t,X_t), \quad \mathbb{P}\text{-a.s.} \quad \text{for all } t\in[0,T]. \tag{3.14}$$

2. *$v \in \mathcal{D}(\mathbb{P})$ and $a^{\mathbb{P}}(v) = fv$. In particular $\mathcal{D}(\mathbb{P})$ is non trivial.*

Proof. 1. Combining (3.10) and (3.11), for all $t\in[0,T[$, $\mathbb{P}$-a.s. we have

$$\begin{aligned} v(t,X_t) &= D_t\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_0^T f(r,X_r)dr - g(X_T)\right)\right]\exp\left(\int_0^t f(r,X_r)dr\right) \\ &= D_t\mathbb{E}^{\mathbb{P}}(v(0,X_0))\exp\left(\int_0^t f(r,X_r)dr\right) = V_t, \end{aligned} \tag{3.15}$$

which shows (3.14).

2. According to Definition 3.4, it will be enough to prove that

$$M_t := V_t - V_0 - \int_0^t f(r,X_r)V_r dr, \; t\in[0,T], \tag{3.16}$$

belongs to $\mathcal{H}^2_{loc}(\mathbb{P})$. Now, by integration by parts, using (3.13) and (3.16), we easily obtain that

$$M_t = \int_0^t \mathbb{E}(V_0)\exp(U^0_r)dD_r, \; t\in[0,T]. \tag{3.17}$$

Consequently M is a stochastic integral w.r.t. the martingale D, hence M is a local martingale. Moreover, since it is the sum of a bounded process and a continuous adapted process (hence locally bounded), M actually belongs to $\mathcal{H}^2_{loc}(\mathbb{P})$.

□

Let us consider again the càdlàg process V defined in (3.13), which is a modification of $(v(t, X_t))$ by Proposition 3.10. The characterization of the probability measure $\mathbb{Q}^*$ defined in (3.8), which is equivalent to the reference probability measure $\mathbb{P}$, as solution of a martingale problem naturally requires the use of Girsanov's theorem. Notice first that $D \in \mathcal{H}^2_{loc}(\mathbb{P})$ since it is a bounded martingale taking into account (3.9) and Hypothesis 3.1. In particular, for any $\mathcal{M} \in \mathcal{H}^2_{loc}(\mathbb{P})$, $\langle \mathcal{M}, D\rangle$ is well-defined under $\mathbb{P}$. Let us then recall the Girsanov's theorem in our context, see for example Theorem 3.11, Chapter III in [26] along with Proposition 3.5 item (i), Chapter III in [26] for the positivity of D.

Theorem 3.11. *(Girsanov). Let $\mathcal{M} \in \mathcal{H}^2_{loc}(\mathbb{P})$. Under $\mathbb{Q}^*$ the process D is strictly positive (up to indistinguishability) and the process $\mathcal{M} - \int_0^\cdot \frac{1}{D_{r-}} d\langle \mathcal{M}, D\rangle_r$ is a $\mathbb{Q}^*$-local martingale.*

About the martingale problem verified by $\mathbb{Q}^*$, we need to specify the linear operator $a^{\mathbb{Q}^*}$ and its domain. For the moment let $\phi \in \mathcal{D}(\mathbb{P})$ and let us apply then Theorem 3.11 to $\mathcal{M} = M[\phi]$ so that we need to compute the bracket $\langle M[\phi], D\rangle$ in order to find the expression of $a^{\mathbb{Q}^*}(\phi)$. This, by (3.17) with $M = M[v]$, is equivalent to computing the bracket $\langle M[\phi], M[v]\rangle$. Indeed, since D is strictly positive $\mathbb{P}$-a.s., the same holds for V by (3.13). By (3.17) we have

$$\langle M[\phi], M[v]\rangle = \int_0^\cdot \mathbb{E}^{\mathbb{P}}[V_0] \exp(U_r^0) d\langle M[\phi], D\rangle_r,$$

so that

$$\langle M[\phi], D\rangle = \int_0^\cdot \frac{\exp(-U_r^0)}{\mathbb{E}^{\mathbb{P}}[V_0]} d\langle M[\phi], D\rangle,$$

consequently

$$\int_0^\cdot \frac{1}{D_{r-}} \langle M[\phi], D\rangle_r = \int_0^\cdot \frac{\exp\left(-U_r^0\right)}{\mathbb{E}^{\mathbb{P}}[V_0] D_{r-}} d\langle M[\phi], M[v]\rangle_r = \int_0^\cdot \frac{d\langle M[\phi], M[v]\rangle_r}{V_{r-}} \tag{3.18}$$

also taking into account (3.13).

Assume now for a moment that $\phi v \in \mathcal{D}(\mathbb{P})$. Since $v \in \mathcal{D}(\mathbb{P})$ and $V_r = v(r, X_r),\ dr \otimes d\mathbb{P}$-a.e., by Proposition 3.10, (3.18) becomes

$$\int_0^\cdot \frac{1}{D_{r-}} \langle M[\phi], D\rangle_r = \int_0^\cdot \frac{\Gamma(\phi, v)(r, X_r)}{V_{r-}} dr = \int_0^\cdot \frac{d\langle M[\phi], M[v]\rangle_r}{V_r} = \int_0^\cdot \frac{\Gamma(\phi, v)(r, X_r)}{v(r, X_r)} dr, \tag{3.19}$$

where $\Gamma(\phi, v)$ is given by (3.4) with $\psi = v$, see Lemma 3.7. Combining Theorem 3.11 with the previous equality, a natural candidate for the operator $a^{\mathbb{Q}^*}$ of the martingale problem verified by $\mathbb{Q}^*$ would be $a^{\mathbb{Q}^*}(\phi) = a^{\mathbb{P}}(\phi) + \frac{\Gamma(\phi,v)}{v}$ for any $\phi \in \mathcal{D}(\mathbb{P})$ verifying $\phi v \in \mathcal{D}(\mathbb{P})$. In view of what precedes, we introduce the following *Ideal Condition* fulfilled by a probability measure $\mathbb{P}$ and a linear subspace $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$.

Condition 3.12. *(Ideal Condition). The element v of L^0 characterized by* (3.11) *is an element of $\mathcal{D}(\mathbb{P})$, such that $\phi v \in \mathcal{D}(\mathbb{P})$ for all $\phi \in \mathcal{D}$.*

Remark 3.13. *The Ideal Condition will be later verified essentially for any subalgebra $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$, under the general hypothesis that $\mathbb{P}$ is Regularly Markovian, see Definition 5.7. This will be the object of Corollary 5.13. Alternatively, in Appendix A, we show the validity of the Ideal Condition under the existence of a $\mathcal{C}^{0,1}$-solution of a PDE, i.e.* (A.3). *Therein we do not suppose that $\mathbb{P}$ is Regularly Markovian.*

Setting $v = \psi$ in Lemma 3.7, we can formulate an alternative formulation of the Ideal Condition 3.12.

Corollary 3.14. *Let $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$ a linear subspace. The following statements are equivalent.*

1. *$(\mathbb{P}, \mathcal{D})$ fulfills the Ideal Condition.*
2. *There exists a linear operator $\Gamma^v : \mathcal{D} \to L^0$, such that for all $\phi \in \mathcal{D}$,*

$$\langle M[\phi], M[v] \rangle = \int_0^{\cdot} \Gamma^v(\phi)(r, X_r) dr. \tag{3.20}$$

Moreover

$$\Gamma^v(\phi) = a^{\mathbb{P}}(v\phi) - va^{\mathbb{P}}(\phi) - \phi a^{\mathbb{P}}(v). \tag{3.21}$$

Proof. This result is a direct consequence of Lemma 3.7, the only thing to check being the linearity of Γ^v which immediately follows from (3.21) as $a^{\mathbb{P}}$ is linear. □

We can then state the kernel result of the paper.

Theorem 3.15. *Let our reference measure $\mathbb{P}$ verify Hypothesis 3.3 (Markov property). Let μ be the law of X_0 and $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$ be a linear subspace of $\mathcal{D}(\mathbb{P})$ such that $(\mathbb{P}, \mathcal{D})$ verifies the Ideal Condition 3.12 and $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ characterized by* (3.11)*. Let $\mathbb{Q}^*$ be the probability measure defined in* (3.8)*.*

Then $\mathbb{Q}^*$ is solution to the martingale problem associated to $(\mathcal{D}, a^{\mathbb{Q}^*}, \nu)$ in the sense of Definition 2.6, where, for all $\phi \in \mathcal{D}$,

$$a^{\mathbb{Q}^*}(\phi)(t,x) := a^{\mathbb{P}}(\phi)(t,x) + \frac{\Gamma^v(\phi)(t,x)}{v(t,x)} \tag{3.22}$$

$$\nu(dx) := v(0,x)\mu(dx) / \int_{\mathbb{R}^d} v(0,y)\mu(dy). \tag{3.23}$$

Remark 3.16. *In particular $\mathcal{D} \subset \mathcal{D}(\mathbb{Q}^*)$.*

Remark 3.17. *As we mentioned in the introduction our Theorem 3.15 has some similarities with Theorem 4.2 from [34] which supposes the existence of a (so-called) "good function" (according to Section 1 in [34]) $v : \mathbb{R}^d \mapsto \mathbb{R}_+^*$, which in particular belongs to $\mathcal{D}(\mathbb{P})$.*

1. *Our Theorem 3.15 implies the result of Theorem 4.2 in [34] under their assumption, at least under, the technical hypotheses on $f := a^{\mathbb{P}}(v)/v$ and $g := -\log(v)$ to be lower bounded. In this case we are in position to apply our Theorem 3.15, which entails the statement of Theorem 4.2 in [34]. Indeed, the particular assumption "$\mathcal{D}_A^h = \mathcal{D}(A)$" of Theorem 4.2 in [34] implies the validity of our Ideal Condition for $(\mathbb{P}, \mathcal{D})$ when $\mathcal{D}$ is the whole extended domain $\mathcal{D}(\mathbb{P})$.*
2. *Theorem 4.2 in [34] (stated in the time inhomogeneous setting) can be used to prove our Theorem 3.15. If we assume that the process in [34] is of the form (t, X_t) with time horizon T and X being an inhomogeneous Markov process, considerations just above* (3.11) *automatically provide the existence of a good function v on the basis of a running cost f and a terminal cost g. That function v is made explicit in Section 5 when $\mathbb{P}$ is regularly Markovian, where the Ideal Condition is always verified.*

Proof of Theorem 3.15. We first check item 1. of Definition 2.6. Let $\psi \in \mathcal{B}_b(\mathbb{R}^d, \mathbb{R})$. Then, taking into account (3.13) and (3.14),

$$\mathbb{E}^{\mathbb{Q}^*}[\psi(X_0)] = \mathbb{E}^{\mathbb{P}}[D_0\psi(X_0)] = \frac{1}{\mathbb{E}^{\mathbb{P}}[v(0,X_0)]}\mathbb{E}^{\mathbb{P}}[v(0,X_0)\psi(X_0)].$$

Hence $\mathcal{L}^{\mathbb{Q}^*}(X_0) = \nu$, where ν is defined in (3.23). It remains to check item 2. of Definition 2.6. Let $\phi \in \mathcal{D}$. Theorem 3.11 states that under $\mathbb{Q}^*$ the process $M[\phi] - \int_0^\cdot \frac{1}{D_{r-}} d\langle M[\phi], D\rangle_r$ is a local martingale. Now by (3.19),

$$\int_0^\cdot \frac{1}{D_{r-}} d\langle M[\phi], D\rangle_r = \int_0^\cdot \frac{\Gamma^v(\phi)(r, X_r)}{v(r, X_r)} dr,$$

where Γ^v is given by Corollary 3.14, and under $\mathbb{Q}^*$ the process

$$\begin{aligned}\phi(\cdot, X_\cdot) - \phi(0, X_0) - \int_0^\cdot a^{\mathbb{P}}(\phi)(r, X_r)dr - \int_0^\cdot \frac{\Gamma^v(\phi)(r, X_r)}{v(r, X_r)} dr \\ = \phi(\cdot, X_\cdot) - \phi(0, X_0) - \int_0^\cdot a^{\mathbb{Q}^*}(\phi)(r, X_r)dr\end{aligned}$$

is a local martingale. This concludes the proof. □

Of course two peculiar cases arises in the following, which will be explored in the sequel.

Hypothesis 3.18. *Let $b \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R}^{d\times d})$. Let $L : [0,T] \times \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d)$ be a Lévy kernel in the sense of Definition 2.1. We fix a truncation function $k : \mathbb{R}^d \to \mathbb{R}^d$, i.e. a bounded real function defined on $\mathbb{R}^d$ equal to the identity in a neighborhood of zero.*

1. *We suppose that $\mathbb{P}$ is solution to the martingale problem with respect to $(\mathcal{D}, a, \mu)$, where $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\mathcal{D} := \mathcal{D}(a) := C_b^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ and $a(\phi)$ is given as*

$$\begin{aligned}a(\phi)(t,x) = \partial_t\phi(t,x) + \langle \nabla_x\phi(t,x), b(t,x)\rangle + \frac{1}{2}Tr[\sigma\sigma^\top(t,x)\nabla_x^2\phi(t,x)] \\ + \int_{\mathbb{R}^d} (\phi(t,x+q) - \phi(t,x) - \langle \nabla_x\phi(t,x), k(q)\rangle) L(t,x,dq),\end{aligned} \tag{3.24}$$

   for all $\phi \in \mathcal{D}$.
2. *$(\mathbb{P}, \mathcal{D})$ verifies previous item 1. replacing the non-local operator* (3.24) *with*

$$a(\phi)(t,x) = \partial_t\phi(t,x) + \langle \nabla_x\phi(t,x), b(t,x)\rangle + \frac{1}{2}Tr[\sigma\sigma^\top(t,x)\nabla_x^2\phi(t,x)], \tag{3.25}$$

   for all $\phi \in \mathcal{D}$.

Remark 3.19. *Let assume Hypothesis 3.18 2. Theorem 2.42, Chapter II in [26] implies that, for all $\phi \in C_b^2(\mathbb{R}^d)$ the process*

$$\begin{aligned}\phi(X_\cdot) - \phi(X_0) - \int_0^\cdot (\nabla_x\phi)^\top(X_r)b(r,X_r)dr - \frac{1}{2}\int_0^\cdot Tr[\sigma\sigma^\top(r,X_r)\nabla_x^2\phi(X_r)]dr \\ - \int_0^\cdot \int_{\mathbb{R}^d} (\phi(X_{r-} + y) - \phi(X_r) - \langle \nabla_x\phi(X_r), y\rangle k(y))\nu^{X,\mathbb{P}}(dr, dy),\end{aligned}$$

is a local martingale under $\mathbb{P}$. By (3.25)*, since the characteristics are uniquely determined, we have $\nu^{X,\mathbb{P}} = 0$. This implies that the jump measure μ^X vanishes and the process X is $\mathbb{P}$-a.s. continuous.*

4. Extension of the Carré du Champ under $\mathbb{P}$

In this Section 4 we further characterize the operator Γ^v appearing in Theorem 3.15 and introduced in Corollary 3.14. Let now $\mathbb{P}$ be our reference probability measure and $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function verifying (3.11) such that $(\mathbb{P}, \mathcal{D})$ fulfills the Ideal Condition 3.12. We start by an assumption on the $\mathbb{P}$-compensator of the jump measure μ^X.

Hypothesis 4.1. *(Compensator). The $\mathbb{P}$-compensator $\nu^{X,\mathbb{P}}$ of the jump measure μ^X of X verifies $\nu^{X,\mathbb{P}}(X, dt, dq) = dtL(t, X_{r-}, dq)$, where L is a Lévy kernel in the sense of Definition 2.1.*

Lemma 4.2. *Assume that $\mathbb{P}$ verifies Hypothesis 4.1. Let $\phi \in \mathcal{D}(\mathbb{P})$. Let Φ be the càdlàg modification of $\phi(\cdot, X_\cdot)$. Then $(\Phi_{t-})_{t\in[0,T]}$ and $(\phi(t, X_{t-}))_{t\in[0,T]}$ are $\mathbb{P}$-indistinguishable.*

Proof. In this proof we make use of the notion of quasi-left continuous processes and predictable stopping time, see Definition 4.3 below.

Let τ be a predictable stopping time. By Theorem 86, Chapter IV in [16], it will be enough to prove

$$\Phi_{\tau-} = \phi(\tau, X_{\tau-}) \quad \mathbb{P}\text{-a.s.} \tag{4.1}$$

on $\{\tau < +\infty\}$. We write

$$\Phi_t = M[\phi]_t + \phi(0, X_0) + \int_0^t a^{\mathbb{P}}(\phi)(r, X_r)dr, \ t \in [0, T]. \tag{4.2}$$

Let now $(\mathcal{S}_n)_{n\geq 1}$ be a localizing sequence for $M[\phi]$ verifying $\int_0^{T\wedge\mathcal{S}_n} |a^{\mathbb{P}}(\phi)(r, X_r)|dr \leq n$. It will be sufficient to prove

$$\Phi_{\tau-}1_{\{\tau<\mathcal{S}_n\}} = \phi(\tau, X_{\tau-})1_{\{\tau<\mathcal{S}_n\}} \quad \mathbb{P}\text{-a.s.} \tag{4.3}$$

To prove what precedes, on the one hand, taking the limit $t \to \tau \wedge \mathcal{S}_n$ from the left in (4.2) yields

$$\Phi_{(\tau\wedge\mathcal{S}_n)-} = M[\phi]_{(\tau\wedge\mathcal{S}_n)-} + \phi(0, X_0) + \int_0^{\tau\wedge\mathcal{S}_n} a^{\mathbb{P}}(\phi)(r, X_r)dr. \tag{4.4}$$

On the other hand, by 1.17, Chapter I in [26], $\{\tau < \mathcal{S}_n\} \in \mathcal{F}_{\tau-}$ for all $n \geq 1$. Setting $X = M[\phi]^{\mathcal{S}_n}$ in Lemma 2.27, Chapter I in [26], by the same Lemma we have that

$$M[\phi]_{(\tau\wedge\mathcal{S}_n)-} = \mathbb{E}^{\mathbb{P}}\left[M[\phi]_{\tau\wedge\mathcal{S}_n}|\mathcal{F}_{\tau-}\right] \text{ on } \{\tau < \mathcal{S}_n\}. \tag{4.5}$$

As $\int_0^{T\wedge\mathcal{S}_n} |a^{\mathbb{P}}(\phi)(r, X_r)|dr \leq n \in L^1(\mathbb{P})$, by (4.5) and (4.2) evaluated at $t = \tau \wedge \mathcal{S}_n$ we get

$$\begin{aligned}
M[\phi]_{(\tau\wedge\mathcal{S}_n)-} &= \mathbb{E}^{\mathbb{P}}\left[M[\phi]_{\tau\wedge\mathcal{S}_n}|\mathcal{F}_{\tau-}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\Phi_{\tau\wedge\mathcal{S}_n}|\mathcal{F}_{\tau-}\right] - \phi(0, X_0) - \int_0^{\tau\wedge\mathcal{S}_n} a^{\mathbb{P}}(\phi)(r, X_r)dr \quad \text{on} \quad \{\tau < \mathcal{S}_n\}.
\end{aligned}$$

Replacing $M[\phi]_{(\tau\wedge\mathcal{S}_n)-}$ in (4.4) we get

$$\Phi_{(\tau\wedge\mathcal{S}_n)-} = \mathbb{E}^{\mathbb{P}}\left[\Phi_{\tau\wedge\mathcal{S}_n}|\mathcal{F}_{\tau-}\right] \quad \text{on} \quad \{\tau < \mathcal{S}_n\},$$

that is

$$\Phi_{\tau-}1_{\{\tau<\mathcal{S}_n\}} = \mathbb{E}^{\mathbb{P}}\left[\phi(\tau, X_\tau)|\mathcal{F}_{\tau-}\right]1_{\{\tau<\mathcal{S}_n\}} \quad \mathbb{P}\text{-a.s.}, \tag{4.6}$$

Hypothesis 4.1 implies that $\nu^{X,\mathbb{P}}(X, \{t\} \times \mathbb{R}^d) = 0$ identically. By Corollary 1.19, Chapter II in [26], the process X is quasi-left continuous under $\mathbb{P}$ in the sense of Definition 4.3 and we have $\Delta X_\tau = 0$ $\mathbb{P}$-a.s. Hence $\phi(\tau, X_\tau) = \phi(\tau, X_{\tau-})$ $\mathbb{P}$-a.s. Moreover, since τ is $\mathcal{F}_{\tau-}$-measurable by 1.14, Chapter I of [26], then $\phi(\tau, X_{\tau-})$ is $\mathcal{F}_{\tau-}$-measurable. Consequently (4.6) then yields

$$\Phi_{\tau-}1_{\{\tau<\mathcal{S}_n\}} = \phi(\tau, X_{\tau-})1_{\{\tau<\mathcal{S}_n\}} \quad \mathbb{P}\text{-a.s.} \tag{4.7}$$

and therefore (4.3). This concludes the proof.

□

Definition 4.3. *The definition below is Definition 2.25 in Chapter I in [26].*

1. *A càdlàg process X is called quasi-left continuous if $\Delta X_\tau = 0$ a.s. on $\{\tau < +\infty\}$ for all predictable stopping times τ, see Definition 2.25 in Chapter I in [26].*
2. *We recall that the notion of predictable stopping time is defined for instance in Definition 2.7, Chapter I in [26].*

Proposition 4.4. *Let $\mathbb{P} \in \mathcal{P}(\Omega)$ and let $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$ be a linear subalgebra. Assume that $(\mathbb{P}, \mathcal{D})$ verifies the Ideal Condition 3.12 and that $\mathbb{P}$ satisfies Hypothesis 4.1. Given Γ^v by* (3.21)*, we define the linear operator $\Gamma^{v,c} : \mathcal{D} \to \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R})$ by*

$$\Gamma^{v,c}(\phi)(t,x) := \Gamma^v(\phi)(t,x) - \int_{\mathbb{R}^d} (v(t,x+q) - v(t,x))(\phi(t,x+q) - \phi(t,x)) L(t,x,dq), \tag{4.8}$$

for $\phi \in \mathcal{D}$. Then

$$[M[v]^c, M[\phi]^c] = \int_0^\cdot \Gamma^{v,c}(\phi)(r, X_r) dr \quad \mathbb{P}\text{-a.s.} \tag{4.9}$$

Remark 4.5. *The identity* (4.9) *also shows that $\Gamma^{v,c}$ can be considered as a map $\mathcal{D} \to L^0$.*

Proof of Proposition 4.4. Let $\phi \in \mathcal{D}$. Using the notation (3.2), let $M[\phi]$ and $M[v]$ be the càdlàg local martingales, which belong to $\mathcal{H}^2_{loc}(\mathbb{P})$, also taking into account Proposition 3.10. Hence $[M[v], M[\phi]] \in \mathcal{A}_{loc}(\mathbb{P})$ by Proposition 4.51, Chapter I in [26], which, taking into account

$$[M[v], M[\phi]] = [M[v]^c, M[\phi]^c] + [M[v]^d, M[\phi]^d], \tag{4.10}$$

yields $[M[v]^d, M[\phi]^d] \in \mathcal{A}_{loc}(\mathbb{P})$.

By Theorem 4.52, Chapter I in [26], we have

$$[M[v]^d, M[\phi]^d] = \sum_{0<r\leq \cdot} \Delta M[v]_r \Delta M[\phi]_r = \sum_{0<r\leq \cdot} \Delta V_r \Delta \Phi_r, \tag{4.11}$$

where V and Φ are càdlàg versions of $v(\cdot, X_\cdot)$ and $\phi(\cdot, X_\cdot)$ respectively. By Lemma 4.2, $\mathbb{P}$-a.s., for all $r \in [0,T]$,

$$\Delta V_r = V_r - V_{r-} = v(r, X_r) - v(r, X_{r-}) = v(r, X_{r-} + \Delta X_r) - v(r, X_{r-})$$

and

$$\Delta \Phi_r = \Phi_r - \Phi_{r-} = \phi(r, X_r) - \phi(r, X_{r-}) = \phi(r, X_{r-} + \Delta X_r) - \phi(r, X_{r-}).$$

Therefore equality (4.11) a.s. gives

$$\begin{aligned} [M[v]^d, M[\phi]^d] &= \sum_{0<r\leq \cdot} (v(r, X_{r-} + \Delta X_r) - v(r, X_{r-}))(\phi(r, X_{r-} + \Delta X_r) - \phi(r, X_{r-})) \\ &= \int_{]0,\cdot] \times \mathbb{R}^d} (v(r, X_{r-} + q) - v(r, X_{r-}))(\phi(r, X_{r-} + q) - \phi(r, X_{r-})) \mu^X(dr, dq). \end{aligned} \tag{4.12}$$

Since the left-hand and right-hand side of previous equality are càdlàg, they are also indistinguishable. Since $[M[v]^d, M[\phi]^d] \in \mathcal{A}_{loc}(\mathbb{P})$ we have

$$\int_{]0,\cdot] \times \mathbb{R}^d} W(r, X_{r-}, q) \mu^X(dr, dq) - \int_{]0,\cdot] \times \mathbb{R}^d} W(r, X_{r-}, q) \nu^{X,\mathbb{P}}(dr, dq) \tag{4.13}$$

is a local martingale, where $W(t,x,q) := (v(t,x+q) - v(t,x))(\phi(t,x+q) - \phi(t,x))$. Consequently $[M[v]^d, M[\phi]^d] - \int_{]0,\cdot] \times \mathbb{R}^d} W(r, X_{r-}, q) \nu^{X,\mathbb{P}}(dr, dq)$ is a local martingale. Since the process $\int_{]0,\cdot] \times \mathbb{R}^d} W(r, X_{r-}, q) \nu^{X,\mathbb{P}}(dr, dq)$ is predictable, we have that

$$\langle M[v]^d, M[\phi]^d \rangle = \int_{]0,\cdot] \times \mathbb{R}^d} W(r, X_{r-}, q) \nu^{X,\mathbb{P}}(dr, dq) \quad \mathbb{P}\text{-a.s.} \tag{4.14}$$

Let $\Gamma^{v,c}$ be the linear operator defined in (4.8). Now (4.10) implies $\langle M[v], M[\phi]\rangle = [M[v]^c, M[\phi]^c] + \langle M[v]^d, M[\phi]^d\rangle$. By (3.20) and (4.14), we have that

$$\int_0^{\cdot} \Gamma^v(\phi)(r, X_r)dr = [M[v]^c, M[\phi]^c] + \int_{]0,\cdot]\times\mathbb{R}^d} W(r, X_{r-}, q)\nu^{X,\mathbb{P}}(dr, dq) \quad \mathbb{P}\text{-a.s.},$$

which immediately yields

$$\int_0^{\cdot} \Gamma^{v,c}(\phi)(r, X_r)dr = [M[v]^c, M[\phi]^c] \quad \mathbb{P}\text{-a.s.}$$

This concludes the proof. □

Proposition 4.4 states that, under Hypothesis 4.1, the operator Γ^v can be decomposed into a jump part and a continuous component $\Gamma^{v,c}$ given by (4.8). We now extend this new operator $\Gamma^{v,c}$ from $\mathcal{D}$ (being a domain of a martingale problem) to a wider domain $\mathbb{D}$. By assumption, for every $\phi \in \mathcal{D}$, $\phi(\cdot, X.)$ is a special semimartingale. This will allow to identify a unique decomposition for $\phi(\cdot, X.)$ even for an important class of non-special semimartingales.

The extension of $\Gamma^{v,c}$ will naturally intervene in the formulation of the martingale problem verified by $\mathbb{Q}^*$ in the examples in Section 6 when the process X is not a special semimartingale under $\mathbb{P}$. For $\phi, \psi \in \mathcal{D}(\mathbb{P})$, let d_c be defined by

$$d_c(\phi, \psi) := \mathbb{E}^{\mathbb{P}}\left[\frac{[M[\phi]^c - M[\psi]^c]_T}{1 + [M[\phi]^c - M[\psi]^c]_T}\right]. \tag{4.15}$$

Remark 4.6. *1. The application d_c introduced by* (4.15) *is a semidistance in the sense that it is non-negative, symmetric, verifies the triangular inequality but $d_c(\phi, \psi)$ might be* 0 *even if $\phi \neq \psi$.*

2. d_c is homogeneous in the sense that $d_c(\phi, \psi) = d_c(\phi - \psi, 0)$ for all $\phi, \psi \in \mathcal{D}(\mathbb{P})$.

We endow L^0 defined in Notation 3.5, with the natural metric

$$d_{L^0}(\phi, \psi) := \mathbb{E}^{\mathbb{P}}\left[\frac{\int_0^T |\phi - \psi|(r, X_r)dr}{1 + \int_0^T |\phi - \psi|(r, X_r)dr}\right]. \tag{4.16}$$

Definition 4.7. *(Closure of $\mathcal{D}$). A linear metric space $(\mathbb{D}, d_{\mathbb{D}})$, where $d_{\mathbb{D}}$ is a homogeneous distance, is said to be a closure of $\mathcal{D}$ if the following holds.*

1. *$\mathcal{D}$ is dense in $\mathbb{D}$ with respect to the metric $d_{\mathbb{D}}$.*
2. *$d_c + d_{L^0} < d_{\mathbb{D}}$ on $\mathcal{D}$.*
3. *The map $\phi \mapsto [\phi(\cdot, X.), N]$ is continuous from $\mathcal{D}$ to $\mathbb{D}^{ucp}$ with respect to the metric $d_{\mathbb{D}}$ for all continuous $\mathbb{P}$-local martingales.*

Remark 4.8. *The following statements are equivalent, given a continuous $\mathbb{P}$-local martingale N.*

1. *$\phi \mapsto [\phi(\cdot, X.), N]$ is continuous with respect to $d_{\mathbb{D}}$.*
2. *$\phi \mapsto [\phi(\cdot, X.), N]$ is continuous in zero.*
3. *For all $\phi \in \mathbb{D}$, $\phi \mapsto [\phi(\cdot, X.), N]$ exists.*

1 ⇔ 2 *is obvious, whereas* 2. ⇔ 3. *follows from the Banach-Steinhaus theorem for F-spaces, see e.g. Chapter 2.1 in [18], taking into account Definition 2.2 item 1.*

Proposition 4.9. *Let $(\mathbb{D}, d_{\mathbb{D}})$ be a closure of $\mathcal{D}$ in the sense of Definition 4.7. Let $\phi \in \mathbb{D}$. Then $\phi(\cdot, X.)$ is a weak Dirichlet process in the sense of Definition 2.2.*

Proof. Let $(\phi_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{D}$ such that $d_{\mathbb{D}}(\phi_n,\phi) \underset{n\to+\infty}{\longrightarrow} 0$. In particular, $d_c(\phi_n,\phi_m) \underset{n,m\to+\infty}{\longrightarrow} 0$, that is $[M[\phi_n]^c - M[\phi_m]^c]_T \overset{\mathbb{P}}{\underset{n,m\to+\infty}{\longrightarrow}} 0$. We consider the unique special semimartingale decomposition

$$\phi_n(\cdot,X_\cdot) = M[\phi_n]^c + M[\phi_n]^d + \int_0^\cdot a^{\mathbb{P}}(\phi_n)(r,X_r)dr =: M[\phi_n]^c + A(\phi_n),$$

where $M[\phi_n]^c$ (resp. $M[\phi_n]^d$) is a continuous (resp. purely discontinuous) local martingale. By Problem 5.25, Chapter 1 in [27], the sequence $(M[\phi_n]^c)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{D}^{ucp}$. Consequently there exists a continuous process M such that $M[\phi_n]^c \underset{n\to+\infty}{\longrightarrow} M$ $\mathbb{P}$-u.c.p. Since the space of continuous $\mathbb{P}$-local martingales vanishing at 0 is closed under u.c.p. convergence, M is a continuous $\mathbb{P}$-local martingale. Clearly $\phi_n(\cdot,X) \to \phi(\cdot,X_\cdot)$ $dt\otimes d\mathbb{P}$-a.e.

We set $A(\phi) := \phi(\cdot,X_\cdot) - M$. Let N be a continuous $\mathbb{P}$-local martingale. To establish that $\phi(\cdot,X_\cdot)$ is a weak Dirichlet process with decomposition $M + A(\phi)$, we need to prove that

$$[A(\phi),N] = 0. \tag{4.17}$$

(4.17) holds for ϕ replaced by $\phi_n\in\mathcal{D}$ since $A(\phi_n)$ has bounded variation. To establish (4.17), we only need to prove that $\psi\mapsto[A(\psi),N]$ is continuous from $\mathcal{D}$ to $\mathbb{D}^{ucp}$. This follows because $\psi\mapsto[\psi(\cdot,X_\cdot),N]$ is continuous by assumption and $\psi\mapsto[M[\psi]^c,N]$ is also continuous by Kunita-Watanabe inequality, taking into account that $d_c < d_{\mathbb{D}}$.

□

From now on, we will denote $M[\phi]^c$ the unique continuous local martingale of the weak Dirichlet decomposition of $\phi(\cdot,X_\cdot)$, see Proposition 2.3.

Example 4.10. *1. Suppose that $\mathcal{D} = \mathcal{C}_b^{1,2}$. and X such that X is a weakly finite quadratic variation process, for instance*

$$\sup_{0<\varepsilon\leq 1}\int_0^T \frac{|X_{r+\varepsilon}-X_r|^2}{\varepsilon}dr < +\infty,\ \mathbb{P}\text{-a.s.}$$

We also suppose Hypothesis 4.1. Consider $\mathbb{D} = \mathcal{C}^{0,1}$. Obviously $\mathcal{D}$ is dense in $\mathcal{C}^{0,1}$ and $\phi(\cdot,X_\cdot)$ is a special semimartingale, and in particular a weak Dirichlet process, for all $\phi\in\mathcal{D}$. By Theorem 4.3 in [3], under $\mathbb{P}$ the canonical process X is a weak Dirichlet process. Let $X = M^c + A$ be the unique decomposition under $\mathbb{P}$ according to Proposition 2.3. By Theorem 3.37 in [3], the unique continuous local martingale part $M[\phi]^c$ of $\phi(\cdot,X_\cdot)$ is

$$M[\phi]^c = \int_0^\cdot (\nabla_x\phi)^\top(r,X_r)dX_r^c. \tag{4.18}$$

It is clear that $d_{L^0} < d_{\mathbb{D}}$. Concerning d_c, for all $\phi\in\mathcal{D}$, (4.18) *implies*

$$[M[\phi]^c]_T = \left[\int_0^\cdot (\nabla_x\phi)^\top(r,X_r)dX_r^c\right]_T = \sum_{i,j=1}^d \int_0^T \partial_{x_i}\phi(r,X_r)\partial_{x_j}\phi(r,X_r)d[X^{c,i},X^{c,j}]_r.$$

Now if $\phi_n \underset{n\to+\infty}{\longrightarrow} 0$ in $\mathcal{C}^{0,1}$, clearly $[M[\phi]^c]_T \underset{n\to+\infty}{\longrightarrow} 0$ in probability under $\mathbb{P}$ and it follows that $d_c(\phi_n,0) \underset{n\to+\infty}{\longrightarrow} 0$. For any $\phi\in\mathcal{C}^{0,1}$, $\phi(\cdot,X_\cdot)$ being a weak Dirichlet process, taking into account (4.18) *we have*

$$[\phi(\cdot,X_\cdot),N] = [M[\phi]^c,N] = \int_0^\cdot (\nabla_x\phi)^\top(r,X_r)d[X^c,N].$$

So the continuity of $\phi\mapsto[\phi(\cdot,X_\cdot),N]$ immediately follows from the previous equality. We conclude from the above that $\mathbb{D}$ is a closure of $\mathcal{D}$. We remark that Theorems 3.37 and 4.3 in [3] are stated in the one-dimensional framework but they can be easily extended to the multidimensional case.

2. *We assume moreover that X has finite variation and has a discrete number of jumps and that $X_t = \sum_{s\le t}(\Delta X_s)$, $\mathbb{P}$-a.s. We set $\mathcal{D} = \mathcal{C}_b^{0,0}$. Then for all $\phi \in \mathcal{D}$,*

$$\phi(t, X_t) = \sum_{s\le t}(\Delta\phi(s, X_s)),\ \mathbb{P}\text{-a.s.}$$

We have $M[\phi]^c = 0$ and

$$\phi(\cdot, X_\cdot) = M[\phi]^d + \int_0^\cdot a^{\mathbb{P}}(\phi)(r, X_r)dr,$$

where

$$a^{\mathbb{P}}(\phi)(r,x) = \int_{\mathbb{R}^d}(\phi(r, x+q) - \phi(r,q))L_r(x, dq),$$

and $M[\phi]^d$ is a purely discontinuous local martingale. Consider $\mathbb{D} = \mathcal{C}^{0,0}$ so that $d_{\mathbb{D}}$ is the metric of $\mathcal{C}^{0,0}$ compatible with the uniform convergence on compact sets. We remark that d_c, given by (4.15) vanishes so that $d_c + d_{L^0} < d_{\mathbb{D}}$. Let $\phi \in \mathbb{D}$. Since $\phi(\cdot, X_\cdot)$ has bounded variation, $[\phi(\cdot, X_\cdot), N]$ exists for all continuous local martingale N and is equal to 0, see e.g. item $d)$ of Proposition 4.49, Chapter I in [26]. In particular $\phi \mapsto [\phi(\cdot, X_\cdot), N]$ is continuous and $\mathbb{D}$ is a closure of $\mathcal{D}$. The map $\Gamma^c = 0$ by (4.9) and so its extension to $\mathbb{D}$ is also trivially zero.

Before proving the main result of this section, we extend the operator $\Gamma^{v,c}$ introduced in Proposition 4.4 from $\mathcal{D}$ to $\mathbb{D}$.

Proposition 4.11. *Let $\mathcal{D} \subset \mathcal{D}(\mathbb{P})$ a subalgebra and $(\mathbb{D}, d_{\mathbb{D}})$ be a closure of $\mathcal{D}$. Assume that $(\mathbb{P}, \mathcal{D})$ verifies the Ideal Condition 3.12. Assume moreover that $\mathbb{P}$ verifies Hypothesis 4.1 and let $\Gamma^{v,c}$ be the operator given by Proposition 4.4. The operator $\Gamma^{v,c} : \mathcal{D} \to L^0$ extends continuously to $\mathbb{D} \to L^0$: we will keep the notation $\Gamma^{v,c}$ for the extension and we still have*

$$\int_0^\cdot \Gamma^{v,c}(\phi)(r, X_r)dr = [M[v]^c, M[\phi]^c] \quad \mathbb{P}\text{-a.s.} \tag{4.19}$$

Proof. Since $d_{\mathbb{D}}$ is homogeneous, in order to prove the continuity extension property, it is enough to check the continuity of $\Gamma^{v,c}$ in 0. Let then $(\phi_n)_{n\ge1}$ be a sequence of elements of $\mathcal{D}$ such that $d_{\mathbb{D}}(\phi_n, 0) \underset{n\to+\infty}{\longrightarrow} 0$. Previous convergence implies that $d_c(\phi_n, 0) \underset{n\to+\infty}{\longrightarrow} 0$, hence $[M[\phi_n]^c]_T \overset{\mathbb{P}}{\underset{n\to+\infty}{\longrightarrow}} 0$ and up to a subsequence we can assume that

$$[M[\phi_n]^c]_T \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{4.20}$$

By (4.9) in Proposition 4.4,

$$\int_0^t \Gamma^{v,c}(\phi_n)(r, X_r)dr = [M[v]^c, M[\phi_n]^c]_t, \tag{4.21}$$

for all $t \in [0, T]$. By Kunita-Watanabe inequality, for all $0 \le s < t$,

$$|d[M[v]^c, M[\phi_n]^c]|(]s,t]) \le \sqrt{[M[v]^c]_t - [M[v]^c]_s}\sqrt{[M[\phi_n]^c]_t - [M[\phi_n]^c]_s}, \tag{4.22}$$

hence, by Cauchy-Schwarz inequality,

$$\begin{aligned}\sup\left\{\sum_{i=1}^p |d[M[v]^c, M[\phi_n]^c]|(]t_{i-1}, t_i])\right\} &\le \sqrt{\left\{\sum_{i=1}^p([M[v]^c]_{t_{i+1}} - [M[v]^c]_{t_i})\right\}}\sqrt{\left\{\sum_{i=1}^p([M[\phi_n]^c]_{t_{i+1}} - [M[\phi_n]^c]_{t_i})\right\}}\\ &= \sqrt{[M[v]^c]_T}\sqrt{[M[\phi_n]^c]_T},\end{aligned} \tag{4.23}$$

where the supremum is taken over all subdivisions $0 = t_0 < t_1 < \ldots < t_p = T$ of $[0,T]$. (4.23) and (4.20) then imply that $d[M[v]^c, M[\phi_n]^c] \to 0$ in the total variation norm for signed measure on $[0,T]$. This immediately yields by (4.21) that

$$\int_0^T |\Gamma^{v,c}(\phi_n)|(r, X_r) \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.},$$

hence $d_{L^0}(\phi_n, 0) \underset{n\to+\infty}{\longrightarrow} 0$ and the conclusion follows.

Concerning (4.19), let $\phi \in \mathbb{D}$ and (ϕ_n) converging to ϕ in $d_{\mathbb{D}}$. By Remark 3.6, for $t \in [0,T]$ we write

$$\left|[M[v]^c, M[\phi]^c]_t - \int_0^t \Gamma^{v,c}(\phi)(r, X_r)dr\right| \leq [M[v]^c, M[\phi - \phi_n]^c]_t + \left|[M[v]^c, M[\phi_n]]^c]_t - \int_0^t \Gamma^{v,c}(\phi_n)(r, X_r)dr\right|$$

$$+ \int_0^t |\Gamma^{v,c}(\phi_n - \phi)|(r, X_r)dr. \tag{4.24}$$

Using Yamada-Watanabe inequality, the fact that $d_c + d_{L^0} < d_{\mathbb{D}}$, taking the limit on the right-hand side of (4.24), we get the left-hand side is zero.

This concludes the proof of (4.19).

□

Proposition 4.12. *Assume that under $\mathbb{P}$ the canonical process X is a weak Dirichlet process with unique decomposition $X = X^c + A$ given by Proposition 2.3. Assume moreover that $\mathbb{P}$ verifies Hypothesis 4.1 and that $\mathcal{D}$ is dense in $\mathbb{D} = \mathcal{C}^{0,1}$. Then $\mathbb{D}$ is a closure of $\mathcal{D}$ in the sense of Definition 4.7 and for all $\phi \in \mathcal{D}$, $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$\Gamma^v(\phi)(t,x) = \langle \Gamma^{v,c}(id)(t,x), \nabla_x \phi(t,x)\rangle + \int_{\mathbb{R}^d} (v(t,x+q) - v(t,x))(\phi(t,x+q) - \phi(t,x))L(t,x,dq), \tag{4.25}$$

with $\Gamma^{v,c}(id) := (\Gamma^{v,c}(id_i))_{1\leq i\leq d}$, $\Gamma^{v,c}$ being the linear operator given by Proposition 4.11.

Proof. The fact that $\mathbb{D}$ is a closure of $\mathcal{D}$ was the object of Example 4.10 item 1. Concerning the proof of (4.25), we see from (4.8) in Proposition 4.4 that it is enough to show that

$$\Gamma^{v,c}(\phi)(t,x) = \langle \Gamma^{v,c}(id)(t,x), \nabla_x \phi(t,x)\rangle \quad dt \otimes d\mathbb{P}_{X_t}\text{-a.e.} \tag{4.26}$$

Recall that for all $\phi \in \mathbb{D}$, by (4.19), we have

$$\int_0^\cdot \Gamma^{v,c}(\phi)(r, X_r)dr = [M[v]^c, M[\phi]^c]. \tag{4.27}$$

In particular, taking $\phi = id_i$, which is an element of $\mathbb{D}$,

$$\int_0^\cdot \Gamma^{v,c}(id)(r, X_r)dr = [M[v]^c, M[id]^c] = [M[v]^c, X^c], \tag{4.28}$$

where $M[id] := (M[id_i])_{1\leq i\leq d}$. Let $\phi \in \mathcal{D}$. Taking into account the definition of L^0, it is enough to prove that

$$[M[v]^c, M[\phi]^c] = \int_0^\cdot (\nabla_x \phi)^\top (r, X_r)\Gamma^{v,c}(id)(r, X_r)dr. \tag{4.29}$$

Now, since X is a weak Dirichlet process under $\mathbb{P}$, Theorem 3.37 in [3] yields

$$M[\phi]^c = \phi(0, X_0) + \int_0^\cdot (\nabla_x \phi)^\top (r, X_r)dX_r^c. \tag{4.30}$$

Finally

$$[M[v]^c, M[\phi]^c] = \int_0^\cdot (\nabla_x \phi)^\top (r, X_r)d[M[v]^c, X^c].$$

gives (4.29) using (4.28).

□

5. Ideal Condition in the Regular Markovian framework

In this section we propose a general framework in which the Ideal Condition 3.12 holds. Most of the following definitions are taken from [4] and [6].

Definition 5.1. *(Markov canonical class). Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be a set of probability measures on $(\Omega,\mathcal{F})$ with corresponding expectation operator maps $(\mathbb{E}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ is called a Markov canonical class if $\mathbb{P}^{s,x}(X_r = x, 0\le r\le s) = 1$ for all $(s,x)\in[0,T]\times\mathbb{R}^d$ and for any $s\le t\le u\le T$, $A\in\mathcal{B}(\mathbb{R}^d)$, $x\to\mathbb{P}^{s,x}(A)$ is Borel and*

$$\mathbb{P}^{s,x}(X_u\in A\mid\mathcal{F}_t) = \mathbb{P}^{t,X_t}(A)\quad \mathbb{P}^{s,x}\text{-a.s.} \tag{5.1}$$

We will say that $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ is measurable in time if $(s,x)\to\mathbb{P}^{s,x}(A)$ is Borel for all $A\in\mathcal{F}$.

Remark 5.2. *A consequence of Proposition B.1 in the Appendix is that for every $(s,x), 0\le s\le t\le T, x\in\mathbb{R}^d$, $\mathbb{P}^{s,x}$ verifies Hypothesis 3.3, i.e. the Markov property.*

Remark 5.3. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be a Markov canonical class. Let Z be a random variable. The proof of the following facts can be found in [4], Proposition 3.10.*

1. *We fix $s\in[0,T]$ and assume that $\mathbb{E}^{s,x}[Z]$ is well-defined for all $x\in\mathbb{R}^d$. Then $x\mapsto\mathbb{E}^{s,x}[Z]$ is Borel.*
2. *Assume moreover that $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ is measurable in time and that $\mathbb{E}^{s,x}[Z]$ is well-defined for all $(s,x)\in[0,T]\times\mathbb{R}^d$. Then $(s,x)\mapsto\mathbb{E}^{s,x}[Z]$ is Borel.*

Definition 5.4. *(Zero-potential set). $A\in\mathcal{F}$ is said to be a zero-potential set if for all $(s,x)\in[0,T]\times\mathbb{R}^d$, $\mathbb{P}^{s,x}(A)=0$.*

We refer to [4] and the references therein for more details on the properties of Markov canonical classes. We will need the notion of *Markovian martingale generator* which extends the notion of *extended generator* introduced in [6].

Definition 5.5. *(Markov martingale domain and generator). Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be a Markov canonical class. We say that $\phi\in\mathcal{B}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ belongs to the Markov martingale domain $\mathcal{D}(\mathfrak{a})$ if there exists a (unique up to a zero-potential set) function χ such that the process*

$$M[\phi]^{s,x} := 1_{[s,T]}(\cdot)\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot \chi(r,X_r)dr\right) \tag{5.2}$$

has a càdlàg modification which is a local martingale under $\mathbb{P}^{s,x}$, for all $(s,x)\in[0,T]\times\mathbb{R}^d$. We set

$$\mathfrak{a}(\phi) := \chi,\ \forall\phi\in\mathcal{D}(\mathfrak{a}), \tag{5.3}$$

$\mathfrak{a} := \mathfrak{a}(\mathbb{P}^{s,x})$ is called Markov martingale generator of $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$. $\mathcal{D}^2(\mathfrak{a})$ will denote the set of elements $\phi\in\mathcal{D}(\mathfrak{a})$ such that $M[\phi]^{s,x}\in\mathcal{H}^2_{loc}(\mathbb{P}^{s,x})$ for all $(s,x)\in[0,T]$, whereas $\mathcal{D}^2_{dt}(\mathfrak{a})$ will denote the set of elements $\phi\in\mathcal{D}^2(\mathfrak{a})$ such that $d\langle M[\phi]^{s,x}\rangle\ll dt$ for all $(s,x)\in[0,T]\times\mathbb{R}^d$.

Remark 5.6. *Clearly $\mathcal{D}(\mathfrak{a})$, $\mathcal{D}^2(\mathfrak{a})$ and $\mathcal{D}^2_{dt}(\mathfrak{a})$ are vector spaces.*

As stated in Section 3 we require more regularity on the reference probability measure $\mathbb{P}$.

Definition 5.7. *(Regularly Markovian). A probability measure $\mathbb{P}\in\mathcal{P}(\Omega)$ is said to be Regularly Markovian if there exists a measurable in time Markov canonical class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ and a probability measure $\mu\in\mathcal{P}(\mathbb{R}^d)$ such that $\mathbb{P}=\int_{\mathbb{R}^d}\mathbb{P}^{0,x}\mu(dx)$. The mentioned Markov canonical class will be said associated with $\mathbb{P}$ and μ will be referred as the initial law.*

In the rest of the section, we are working with a Regularly Markovian probability measure $\mathbb{P}$ associated with $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ and initial law μ.

For all $(s,x)\in[0,T]\times\mathbb{R}^d$ we set

$$v(s,x)=\mathbb{E}^{s,x}\left[\exp\left(-\int_s^T f(r,X_r)dr-g(X_T)\right)\right]. \tag{5.4}$$

Lemma 5.8. *$v\in\mathcal{B}_b([0,T]\times\mathbb{R}^d,\mathbb{R})$ and fulfills* (3.11). *Moreover $v\in\mathcal{D}(\mathbb{P})$ and $a^{\mathbb{P}}(v)=fv$.*

Proof. $v\in\mathcal{B}_b([0,T]\times\mathbb{R}^d,\mathbb{R})$ by Remark 5.3. By the Markov property, provided by Lemma B.5 it is clear that the function v given by (5.4) satisfies (3.11). The result then follows from Proposition 3.10. □

Remark 5.9. *v defined in* (5.4) *is independent of the underlying Markov canonical class $(\mathbb{P}^{s,x})$. Indeed, suppose that $\mathbb{P}$ is associated with another Markov canonical class $(\tilde{\mathbb{P}}^{s,x})$. In this case* (5.4) *would characterize another function $\tilde{v}$. Nevertheless we know by Remark 3.5 that $v=\tilde{v}$ in L^0.*

Remark 5.10. *Choosing $\mathbb{P}=\mathbb{P}^{0,x}$, Lemma 5.8 yields that $v\in\mathcal{D}(\mathbb{P}^{0,x})$ so that $M[v]^{s,x}\in\mathcal{H}^2_{loc}(\mathbb{P}^{s,x})$ for all $x\in\mathbb{R}^d$ and $s=0$. Analogously this extends to all $s\in[0,T]$. In particular, v is an element of $\mathcal{D}^2(\mathfrak{a})$.*

By Lemma 5.8 the function v defined by (5.4) verifies (3.11) and constitutes a good candidate to satisfy the Ideal Condition 3.12 with respect to the Regularly Markovian probability measure $\mathbb{P}$. It remains to find a suitable domain $\mathcal{D}\subset\mathcal{D}(\mathbb{P})$ such that $v\phi\in\mathcal{D}(\mathbb{P})$ for all $\phi\in\mathcal{D}$, or equivalently to prove the existence of an operator $\Gamma^v:\mathcal{D}\to L^0$ such that (3.20) holds, see Corollary 3.14. This is the aim of the crucial Proposition 5.11 below, whose proof is postponed to the Appendix C for the sake of clarity.

Proposition 5.11. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be a Markov canonical class. We have the following.*

1. *There exists a bilinear map $\Gamma:\mathcal{D}^2_{dt}(\mathfrak{a})\times\mathcal{D}^2(\mathfrak{a})\to\mathcal{B}([0,T]\times\mathbb{R}^d,\mathbb{R})$ such that for all $\psi\in\mathcal{D}^2(\mathfrak{a})$, $\phi\in\mathcal{D}^2_{dt}(\mathfrak{a})$ and for all $(s,x)\in[0,T]\times\mathbb{R}^d$,*

$$\langle M[\phi]^{s,x},M[\psi]^{s,x}\rangle_\cdot=1_{[s,T]}(\cdot)\int_s^\cdot\Gamma(\phi,\psi)(r,X_r)dr\quad\mathbb{P}^{s,x}\text{-a.s.} \tag{5.5}$$

Let $\mathbb{P}$ be a Regularly Markovian probability measure associated with $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ and initial law μ.

2. *Let v defined in* (5.4). *There exists a linear map $\Gamma^v:\mathcal{D}^2_{dt}(\mathfrak{a})\to L^0$ such that for all $\phi\in\mathcal{D}^2_{dt}(\mathfrak{a})$,*

$$\langle M[\phi],M[v]\rangle=\int_0^\cdot\Gamma^v(\phi)(r,X_r)dr\quad\mathbb{P}\text{-a.s.} \tag{5.6}$$

Remark 5.12. *Given ϕ,ψ, the function $\Gamma(\phi,\psi)$ is uniquely defined up to a zero-potential set. In fact, to be rigorous, this means that the operator Γ takes values in the linear space of classes of Borel functions up to zero potential sets.*

The following corollary is the main result of this section.

Corollary 5.13. *Let $\mathbb{P}$ be a Regularly Markovian probability measure and let $\mathcal{D}\subset\mathcal{D}^2_{dt}(\mathfrak{a})\subset\mathcal{D}(\mathbb{P})$ be a subalgebra, where $\mathcal{D}^2_{dt}(\mathfrak{a})$ is given by Definition 5.5. Then $(\mathbb{P},\mathcal{D})$ verifies the the Ideal Condition 3.12.*

Proof. By Lemma 5.8 v is an L^0-version of the one introduced in (3.11). Moreover, $\phi v\in\mathcal{D}(\mathbb{P})$ for all $\phi\in\mathcal{D}$, by Proposition 5.11 item 2. and Corollary 3.14. □

Remark 5.14. *The function v is a (so called) decoupled mild solution of the Pseudo-Partial Differential Equation*

$$\begin{cases} a(v) = fv \\ v(T,.) = g, \end{cases}$$

in the sense of Definitions 5.3 and 5.13 in [6]. This extends the case illustrated in the Appendix A, where v is either a classical or a strong solution of the PDE (A.3). *In the general case v is a priori only measurable.*

6. Examples of applications

In this section we will provide examples of probability measures $\mathbb{P}$ which are solutions of martingale problem with respect to $(\mathcal{D}, a, \mu)$, for some integro-PDE operators a. In this case $(\mathbb{P}, \mathcal{D})$ will fulfill the Ideal Condition 3.12 since $\mathbb{P}$ is a Regularly Markovian probability measure, associated with peculiar examples of Markov canonical classes.

Remark 6.1. *1. In this section v will be the Borel function defined in* (5.4)*.*

2. *$\Gamma^{v,c} : \mathcal{D} \to L^0$ is the map defined in the sense of Proposition 4.4; the same notation will refer to its extension to the closure on $\mathbb{D} = C^{0,1}$ in Proposition 4.11.*

6.1. Markovian jump diffusions

We focus in this section on the case of Markovian diffusion with jumps, namely when $(\mathbb{P}, \mathcal{D})$ satisfies Hypothesis 3.18 item 1.

Hypothesis 6.2. *1. $\mathbb{P}$ is regularly Markovian in the sense of Definition 5.7*

2. *$\mathcal{D} \subset \mathcal{D}^2_{dt}(\mathfrak{a})$, where the latter set was defined in Definition 5.5*

Remark 6.3. *We list below some cases in which Hypotheses 3.18 and 6.2 are fulfilled.*

- *$b, \sigma, \int_{|q|\le 1} |q|^2 L(\cdot, dq)$ have linear growth uniformly in t, σ is bounded continuous and non-degenerate, see Theorem 5.2 in [28].*
- *There is no diffusion component, b is bounded continuous, $(t,x) \mapsto \int \frac{|q|^2}{1+|q|^2}\varphi(q) L(t,x,dq)$ is continuous for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ and $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |q|^2) L(t,x,dq) < +\infty$, see Theorem 2.2 in [40] for the existence and Theorem 3 in [29] for the uniqueness. This typically includes the case of α-stable Lévy processes, for instance Cauchy processes.*

Proposition 6.4. *Let $(\mathbb{P}, \mathcal{D})$ verifying Hypotheses 3.18 item 1. and 6.2. Set $\mu := \mathcal{L}^{\mathbb{P}}(X_0)$. Let ν be defined in* (3.23) *and v by* (5.4)*. Then the exponential twist $\mathbb{Q}^*$ given by* (3.8) *is solution of a martingale problem $(\mathcal{D}, a^{\mathbb{Q}^*}, \nu)$ and for all $\phi \in \mathcal{D}$*

$$\begin{aligned} a^{\mathbb{Q}^*}(\phi)(t,x) &= a(\phi)(t,x) + \langle \nabla_x \phi(t,x), \Gamma(v)(t,x) \rangle \\ &\quad + \int_{\mathbb{R}^d} (v(t,x+q) - v(t,x))(\phi(t,x+q) - \phi(t,x)) L(t,x,dq), \end{aligned} \tag{6.7}$$

where

$$\Gamma(v)(s,x) := [\Gamma^{v,c}(id_i)(s,x)]_{1\le i\le d}\,, \tag{6.8}$$

where $\Gamma^{v,c}$ is the map introduced in Remark 6.1 2.

Proof. Corollary 5.13 states that $(\mathbb{P}, \mathcal{D})$ fulfills the Ideal Condition 3.12. Notice that $\mathbb{P}$ also verifies Hypothesis 4.1. The result is then a direct application of Theorem 3.15. taking into account Proposition 4.12.

□

6.2. The case of Markovian diffusions

We consider here the particular case of Brownian diffusions, when $(\mathbb{P},\mathcal{D})$ verifies Hypothesis 3.18 item 2.

Remark 6.5. *We emphasize that we do not make any assumption on the coefficients σ, b of the martingale problem. In fact, we do not even require local boundedness of these coefficients. All the results of this paper are based on the properties verified by the probability measure $\mathbb{P}$ (in particular the Regularly Markovian feature) without any restriction on the generator a of the underlying martingale problem.*

Remark 6.6. *Hypothesis 6.2 is verified for instance in the following cases.*

- *σ, b have linear growth and σ is continuous and non-degenerate, see [43] Corollary 7.1.7 and Theorem 10.2.2.*
- *$d = 1$ and σ is lower bounded by a positive constant on each compact set, see [43], Exercise 7.3.3.*
- *$d = 2$, $\sigma\sigma^\top$ is non-degenerate and σ and b are time-homogeneous and bounded, see [43], Exercise 7.3.4.*
- *σ, b are Lipschitz with linear growth (with respect to the space variable, independently in time).*
- *σ, b are bounded continuous, see Chapter 12 in [43] and the Markov selection therein.*

Corollary 6.7. *Let $(\mathbb{P},\mathcal{D})$ verifying Hypothesis 3.18 item 2. and Hypothesis 6.2. Set $\mu := \mathcal{L}^{\mathbb{P}}(X_0)$. Let ν be defined in* (3.23) *and v by* (5.4). *Then the exponential twist $\mathbb{Q}^*$ given by* (3.8) *is solution of a martingale problem $(\mathcal{D}, a^{\mathbb{Q}^*}, \nu)$, where for all $\phi \in \mathcal{D}$*

$$a^{\mathbb{Q}^*}(\phi)(t,x) = \partial_t\phi(t,x) + \left\langle \nabla_x\phi(t,x), b(t,x) + \frac{\Gamma(v)(t,x)}{v(t,x)} \right\rangle + \frac{1}{2}Tr[\sigma\sigma^\top(t,x)\nabla_x^2\phi(t,x)], \tag{6.9}$$

where

$$\Gamma(v)(s,x) := [\Gamma^{v,c}(id_i)(s,x)]_{1\le i\le d}\,, \tag{6.10}$$

where $\Gamma^{v,c}$ is again the map introduced in Remark 6.1 2.

Proof. Since $L \equiv 0$, the probability measure $\mathbb{P}$ verifies Hypothesis 4.1 and the result is a direct application of Proposition 6.4. □

Remark 6.8. *When the function v is an element of $\mathcal{C}^{0,1}$, then Proposition A.1 implies that $\Gamma(v) = \sigma\sigma^\top\nabla_x v$. Indeed, by Lemma 5.8, $a^{\mathbb{P}}(v) = fv$.*

In the more irregular case when $\mathbb{P}$ is only Regularly Markovian, $\Gamma(v)$ can then be interpreted as a generalized gradient, see [6] for more consideration on this notion.

We state now some consequences which will be used in the companion paper [10], when the coefficients b, σ have linear growth. To avoid more technical conditions we will suppose the initial condition to be deterministic, i.e. $\mu = \delta_x$, for some $x \in \mathbb{R}^d$.

Hypothesis 6.9.

1. *The coefficients b, σ in* (3.25) *satisfy*

$$|b(t,x)| + \|\sigma(t,x)\| \le C(1+|x|), \tag{6.11}$$

for some constant $C > 0$.

2. *σ is uniformly elliptic in the sense that, for all $(t,x) \in [0,T]\times\mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\xi^\top\sigma\sigma^\top(t,x)\xi \ge c_\sigma|\xi|^2$, for some constant $c_\sigma > 0$.*

Lemma 6.10. *Assume Hypothesis 6.9. Let $\mathbb{P}$ be a solution of the martingale problem $(\mathcal{D}, a, \delta_x)$ for some $x \in \mathbb{R}^d$ and operator a defined by* (3.25). *Let v be the function defined in* (5.4) *and $\Gamma(v)$ defined in* (6.10). *Then for all $1 < p < 2$,*

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|\frac{\Gamma(v)(r,X_r)}{v(r,X_r)}\right|^p dr\right] < +\infty.$$

Proof. Since Hypothesis 6.9 holds, $(\mathbb{P}, \mathcal{C}_b^{1,2})$ verifies Hypothesis 6.2, see Remark 6.6. Then by Corollary 6.7, under $\mathbb{Q}^*$ the canonical process decomposes into

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \frac{\Gamma(v)(r, X_r)}{v(r, X_r)}dr + M_t^{\mathbb{Q}^*}, \tag{6.12}$$

where $M^{\mathbb{Q}^*}$ is a $\mathbb{Q}^*$-local martingale such that $\langle M^{\mathbb{Q}^*}\rangle_t = \int_0^t \sigma\sigma^\top(r, X_r)dr$. This decomposition is a direct consequence of Proposition 5.4.6 in [27], noticing that the martingale problem verified by $\mathbb{P}$ extends to $\mathcal{D} = C^{1,2}([0,T], \mathbb{R}^d)$. On the other hand, since $H(\mathbb{Q}^*|\mathbb{P}) < +\infty$, Theorem 2.1 in [30] gives the existence of a progressively measurable process α such that

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T |\sigma^\top(r, X_r)\alpha_r|^2 dr\right] < +\infty, \tag{6.13}$$

and under $\mathbb{Q}^*$ the canonical process has decomposition

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma\sigma^\top(r, X_r)\alpha_r dr + \tilde{M}_t, \tag{6.14}$$

where the $\mathbb{Q}^*$-local martingale $\tilde{M}$ verifies $\langle \tilde{M}\rangle_t = \int_0^t \sigma\sigma^\top(r, X_r)dr$. Identifying the bounded variation and the martingale components between (6.12) and (6.14), we get $\tilde{M} = M^{\mathbb{Q}^*}$ and

$$\sigma\sigma^\top(\cdot, X_\cdot)\alpha_\cdot = \Gamma(v)(\cdot, X_\cdot)/v(\cdot, X_\cdot), \ dt \otimes d\mathbb{Q}^* - \text{a.e.} \tag{6.15}$$

Besides, since $\|d\mathbb{Q}^*/d\mathbb{P}\|_\infty < +\infty$, the linear growth of σ and classical moments estimates under $\mathbb{P}$ yield

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \|\sigma(r, X_r)\|^q dr\right] < +\infty, \tag{6.16}$$

for all $q \geq 1$. We then fix $1 < p < 2$. By Holder's inequality applied on the measure space $([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}, dt \otimes d\mathbb{Q}^*)$, it holds that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T |\sigma\sigma^\top(r, X_r)\alpha_r|^p dr\right] &\leq \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \|\sigma(r, X_r)\|^p |\sigma^\top(r, X_r)\alpha_r|^p dr\right] \\ &\leq \left(\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \|\sigma(r, X_r)\|^{2p/(2-p)} dr\right]\right)^{1-p/2} \left(\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T |\sigma^\top(r, X_r)\alpha_r|^2 dr\right]\right)^{p/2}.\end{aligned}$$

Combining previous inequality with (6.13) and (6.16), taking into account (6.15), yields

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|\frac{\Gamma(v)(r, X_r)}{v(r, X_r)}\right|^p dr\right] = \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T |\sigma\sigma^\top(r, X_r)\alpha_r|^p dr\right] < +\infty.$$

□

The corollary below constitutes a key tool in [10].

Corollary 6.11. *Let X be a solution in law of the SDE*

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x, \end{cases}$$

where b, σ verify Hypothesis 6.9. Then there exists a function $\beta \in \mathcal{B}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ such that exponential twist $\mathbb{Q}^*$ given by (3.8) *is the law of a weak solution of the SDE*

$$\begin{cases} dX_t = (b(t,X_t) + \beta(t,X_t))\,dt + \sigma(t,X_t)dW_t \\ X_0 = x. \end{cases}$$

Moreover, $\beta \in L^p([0,T]\times\mathbb{R}^d, dt\otimes d\mathbb{Q}^*)$ for all $1 \le p < 2$.

Proof. Since Hypothesis 6.9 holds, $(\mathbb{P}, \mathcal{C}_b^{1,2})$ verifies Hypothesis 6.2, see Remark 6.6, so that we can apply Corollary 6.7 and Lemma 6.10. The result follows from the equivalence between weak solution of stochastic differential equations (SDE) and solution of martingale problems associated to $(\mathcal{D}, a)$ where a is given by (3.25), see e.g. Proposition 5.4.6 in [27].

□

6.3. SDEs with distributional drift

We apply in this section our result to a more irregular framework where the reference probability measure $\mathbb{P}$ is solution of a martingale problem with parabolic generator $a(\phi) = \partial_t\phi + \langle\nabla_x\phi, b\rangle + \frac{1}{2}\Delta\phi$ where the drift $b = (b^1,\ldots,b^d)$ is a (vector valued Schwartz) distribution. We will use throughout this section the formalism and some of the results from [25]. Let $\mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions with values in $\mathbb{R}$ and let $\mathcal{S}'(\mathbb{R}^d)$ be the space of Schwartz distributions. For $\gamma \in \mathbb{R}$ we denote $\mathcal{C}^\gamma(\mathbb{R}^d)$ the Besov space $\mathcal{B}^\gamma_{\infty,\infty}$. For details on Besov spaces we refer to Section 2.7 in [1]. In particular, for any $\phi \in \mathcal{C}^\alpha(\mathbb{R}^d)$, $\psi \in \mathcal{C}^{-\beta}(\mathbb{R}^d)$ for $\alpha, \beta > 0$ such that $\alpha - \beta > 0$, one can define the pointwise product $\phi\psi \in \mathcal{C}^{-\beta}(\mathbb{R}^d)$. We also define $\mathcal{C}^{\gamma+}(\mathbb{R}^d) := \bigcup_{\alpha>\gamma} \mathcal{C}^\alpha(\mathbb{R}^d)$. $\mathcal{C}^\gamma_c(\mathbb{R}^d)$ will denote the set of elements of $\mathcal{C}^\gamma(\mathbb{R}^d)$ with compact support. Finally we denote $\bar{\mathcal{C}}^\gamma_c(\mathbb{R}^d)$ the space

$$\bar{\mathcal{C}}^\gamma_c(\mathbb{R}^d) := \{\phi \in \mathcal{C}^\gamma(\mathbb{R}^d) \ : \ \exists(\phi_n) \subset \mathcal{C}^\gamma_c(\mathbb{R}^d) \textit{ such that } (\phi_n) \to \phi \textit{ in } \mathcal{C}^\gamma(\mathbb{R}^d)\}$$

and we define the spaces $\bar{\mathcal{C}}^{\gamma+}_c(\mathbb{R}^d)$ as $\mathcal{C}^{\gamma+}(\mathbb{R}^d)$.

Let $0 < \beta < \frac{1}{2}$. According to Theorem 4.5 in [25] there is a unique probability measure $\mathbb{P}$ being solution to the martingale problem (with distributional drift) with respect to $(\mathcal{D}, a, \mu)$, where

$$\begin{aligned} \mathcal{D} := \Big\{ \phi \in C\left([0,T], \mathcal{C}^{(1+\beta)+}(\mathbb{R}^d)\right) \ : \ \exists\varphi \in C\left([0,T], \bar{\mathcal{C}}^{0+}_c(\mathbb{R}^d)\right) \text{ such that} \\ \phi \text{ is a weak solution of } a(\phi) = \varphi \text{ and } \phi(T,.) \in \bar{\mathcal{C}}^{(1+\beta)+}_c(\mathbb{R}^d) \Big\} \end{aligned} \tag{6.17}$$

and $a(\phi) = \partial_t\phi + \langle\nabla_x\phi, b\rangle + \frac{1}{2}\Delta\phi$ for a drift $b \in C\left([0,T], \mathcal{C}^{(-\beta)+}(\mathbb{R}^d)\right)$. We remark that $\langle\nabla_x\phi, b\rangle := \sum_i \partial_{x_i}\phi b^i$ and the products $\partial_{x_i}\phi b^i$ are pointwise products.

Remark 6.12.

1. *The aforementioned probability measure $\mathbb{P}$ is Regularly Markovian since it is the Zvonkin transform of a probability measure fulfilling a martingale problem of the same type as the one in the first bullet point in Remark 6.6, see Theorem 3.9 in [25]. Also, since the martingale problem can be solved for any deterministic initial condition we have $\mathcal{D} \subset \mathcal{D}^2_{dt}(\mathfrak{a})$.*
2. *The martingale problem in [25] is stated on the canonical space of the continuous functions on $C([0,T],\mathbb{R}^d)$ instead on $D([0,T],\mathbb{R}^d)$. However, using similar arguments as in the discussion following Remark 6.6 at the level of the Zvonkin transformed process, one can show that the jump measure is necessarily zero, whenever the martingale problem is formulated in the space of càdlàg functions.*

We are now ready to characterize the solution $\mathbb{Q}^*$ to Problem (1.3) in this framework.

Proposition 6.13. *Let* $(\mathbb{P},\mathcal{D})$ *introduced above. Set* $\mu := \mathcal{L}^{\mathbb{P}}(X_0)$*. Let* ν *be defined in* (3.23) *and* v *by* (5.4)*. Then the exponential twist* $\mathbb{Q}^*$ *given by* (3.8) *is solution of a martingale problem* $(\mathcal{D}, a^{\mathbb{Q}^*}, \nu)$*, where* $\mathcal{D}$ *is given by* (6.17) *and*

$$a^{\mathbb{Q}^*}(\phi) = a(\phi) + \left\langle \nabla_x \phi, \frac{\Gamma(v)}{v} \right\rangle,$$

where $\Gamma(v) := (\Gamma^{v,c}(id_i))_{1\leq i \leq d}$ *is provided by Proposition 4.11.*

Proof. Corollary 5.13 together with Remark 6.12 imply that $(\mathbb{P},\mathcal{D})$ fulfills the Ideal Condition 3.12. Moreover under $\mathbb{P}$ the canonical process X is a continuous weak Dirichlet process by Proposition 5.11 in [25] applied with $f = id$. In particular, $\mathbb{P}$ verifies Hypothesis 4.1 with $L = 0$. Since $\mathcal{D}$ is dense in $\mathcal{C}^{0,1}$ by Lemma 5.7 in [25], we can apply Proposition 4.12 which says that $\mathbb{D} = C^{0,1}$ is a closure of $\mathcal{D}$ and

$$\Gamma^v(\phi) = \langle \nabla_x \phi, \Gamma^{v,c}(id) \rangle, \ \forall \phi \in \mathcal{D}$$

and $\Gamma^{v,c}(id)$ is provided by Proposition 4.11. The result then follows from Theorem 3.15. □

Appendix A: PDE characterization of v: an illustration in the case of Markov diffusions

In the whole section, our reference measure $\mathbb{P}$ will be supposed again to fulfill the Markov property 3.3 but not necessarily, a priori, to be Regularly Markovian. We assume on the other hand that $\mathbb{P}$ fulfills item 2. of Hypothesis 3.18.

We recall that a is given by (3.25). We recall that, in this case $\mathcal{D} = \mathcal{C}_b^{1,2} \subset \mathcal{D}(\mathbb{P})$, see Remark 3.6 item 3.

By Remark 3.19 X is continuous. By obvious extension arguments we can replace $\mathcal{D}$ with $\mathcal{C}^{1,2}$ and by Proposition 4.6 in Chapter 5 of [27], we easily obtain that

$$M[id]_t := X_t - X_0 - \int_0^t b(r, X_r) dr \tag{A.1}$$

is a continuous local martingale satisfying

$$[M[id]]_t = \int_0^t \sigma\sigma^\top(r, X_r) dr. \tag{A.2}$$

Alternatively to the Regularly Markovian property, we focus on the PDE

$$\begin{cases} a(w)(t,x) = w(t,x) f(t,x), \ (t,x) \in [0,T[\times \mathbb{R}^d \\ w(T,.) = e^{-g}. \end{cases} \tag{A.3}$$

The aim of the proposition below is to show the validity of the Ideal Condition provided we find a solution of (A.3).

Proposition A.1. *Suppose there is* $w \in C^{0,1}[0,T] \times \mathbb{R}^d)$ *belonging to* $\mathcal{D}(\mathbb{P})$ *and verifying* $a^{\mathbb{P}}(w) = fw$*. Assume moreover that* w *is bounded. Then* w *is an* L^0*-version of the function* v *which is defined by* (3.11) *Moreover it fulfills the Ideal Condition 3.12 and for all* $\phi \in \mathcal{D}$*, we have*

$$\Gamma^{v,c}(\phi) := (\nabla_x \phi)^\top \sigma\sigma^\top \nabla_x v. \tag{A.4}$$

Proof. By assumption, for all $t \in [0,T]$ we have

$$w(t, X_t) = w(0, X_0) + \int_0^t (fw)(r, X_r) dr + M[w]_t,$$

where $M[w]$ is a local martingale. Then by integration by parts, the process

$$\exp\left(-\int_0^t f(r, X_r) dr\right) w(\cdot, X.)$$

is a local martingale, which is a genuine martingale since it is bounded. Consequently, by taking the conditional expectation with respect to $\mathcal{F}_t$ and making use of the Markov property (3.1) we get

$$w(t,X_t) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\middle| X_t\right] \quad \mathbb{P}\text{-a.s.},$$

for all $t \in [0,T]$. So w verifies (3.11), and by Remark 3.5, $w = v$ in L^0.

It remains to prove that $(\mathbb{P},\mathcal{D})$ fulfills the Ideal Condition 3.12. Since $w \in \mathcal{C}^{0,1}$, by Theorem 3.37 in [3], $v(\cdot,X.)$ is a weak Dirichlet process with local martingale component

$$M[v] = v(0,X_0) + \int_0^{\cdot} (\nabla_x v)^\top (r,X_r) dM[id]_r,$$

where $M[id]$ is given in (A.1). Consequently for all $\phi \in \mathcal{D}$, taking into account (A.2) in the fourth equality

$$\begin{aligned} ,M[v]]_t = [\phi(\cdot,X.), v(\cdot,X)]_t &= \int_0^t (\nabla\phi)^\top (r,X_r) d[X,M[v]]_r \\ &= \int_0^t (\nabla\phi)^\top (r,X_r) d[M[id],M[v]]_r \\ &= \int_0^t (\nabla_x\phi)^\top (r,X_r)\sigma\sigma^\top (r,X_r)\nabla_x v(r,X_r)dr \\ &= \int_0^t \Gamma^{v,c}(\phi)(r,X_r)dr, \end{aligned}$$

where $\Gamma^{v,c}(\phi)$ is given by (A.4). The Ideal Condition then follows by Corollary 3.14 applied to $\psi = v$. □

Example A.2. *We provide below some examples of application of Proposition A.1.*

1. *Suppose that $w \in \mathcal{C}_b^{1,2}$ is a (classical) solution of* (A.3). *A direct application of Itô's formula proves that $a^{\mathbb{P}}(w) = fw = a(w)$ so that we can apply Proposition A.1 which tells that $w = v$ in L^0 and v is a classical solution of* (A.3).
2. *We focus now on the case of strong solutions in the sense of Definition A.3 below.*

   Definition A.3. *A function $w \in \mathcal{B}([0,T]\times\mathbb{R}^d,\mathbb{R})$ is called a strong solution to equation* (A.3) *if there exists a sequence (w_n,f_n,g_n) of Borel functions $w_n \in \mathcal{C}_b^{1,2}$, $f_n \in \mathcal{B}([0,T]\times\mathbb{R}^d,\mathbb{R})$ and $g_n \in \mathcal{B}([0,T]\times\mathbb{R}^d,\mathbb{R})$ verifying the following.*

   (i) w_n is a classical solution of equation

   $$\text{(A.5)} \qquad \begin{cases} a(w_n)(t,x) = f_n(t,x)w_n(t,x), \ (t,x) \in [0,T[\times\mathbb{R}^d \\ w_n(T,.) = e^{-g_n}, \end{cases}$$

   (ii) (w_n,f_n,g_n) converges uniformly on compact sets, towards (w,f,g) as $n \to +\infty$.
3. *We show below, via Proposition A.1, that the Ideal Condition holds when w is a $\mathcal{C}^{0,1}$-strong solution of* (A.3). *Let (w_n,f_n,g_n) be the associated sequence. In this case, $w_n \in \mathcal{D}(\mathbb{P})$ and for all $t \in [0,T]$,*

   $$w_n(t,X_t) = w_n(0,X_0) + \int_0^t (f_n w_n)(r,X_r)dr + M[w_n]_t.$$

   Recall that the space of continuous local martingale is closed under u.c.p. convergence. Letting $n \to +\infty$ in the previous equality in the u.c.p. sense then yields

   $$w(t,X_t) - w(0,X_0) - \int_0^t (fw)(r,X_r)dr = M[w]_t, \ t \in [0,T],$$

where $M[w]$ is a continuous local martingale. Hence $w \in \mathcal{D}(\mathbb{P})$ and $a^{\mathbb{P}}(w) = fw$. Since $w \in \mathcal{C}^{0,1}$ by assumption, Proposition A.1 applies, so that v is a strong solution of (A.3).

Appendix B: Markov property

The objective of the section is to prove that a reference probability $\mathbb{P}$, which is Regularly Markovian verifies the Markov property (Hypothesis 3.3).

Proposition B.1. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be a Markov canonical class in the sense of Definition 5.1. Then for all $0 \le s \le t \le T$, $F \in \mathcal{B}_b(D([t,T],\mathbb{R}^d),\mathbb{R})$ it holds that*

$$\mathbb{E}^{s,x}\left[F((X_u)_{u\in[0,T]}) \mid \mathcal{F}_t\right] = \mathbb{E}^{t,X_t}[F((X_u)_{u\in[0,T]})].$$

We first prove a weaker version of this proposition in order to apply a functional version of the monotone class lemma to prove Proposition B.1.

Lemma B.2. *For all $n \ge 1$, $t \le t_1 \le \ldots \le t_n \le T$, $f \in \mathcal{B}_b((\mathbb{R}^d)^n,\mathbb{R})$,*

$$\mathbb{E}^{s,x}[f(X_{t_1},\ldots,X_{t_n}) \mid \mathcal{F}_t] = \mathbb{E}^{t,X_t}\left[f(X_{t_1},\ldots,X_{t_n})\right] \ \mathbb{P}^{s,x}\text{-}a.s.$$

Proof. Let first $f_1,\ldots,f_n$ belong to $\mathcal{B}_b(\mathbb{R}^d,\mathbb{R}^+)$, $t \le t_1 \le \ldots \le t_n \le T$. Let $B \in \mathcal{F}_t$. We first prove by induction that

$$\mathbb{E}^{s,x}[1_B f_1(X_{t_1})\ldots f_n(X_{t_n})] = \mathbb{E}^{s,x}\left[1_B \mathbb{E}^{t,X_t}\left[f_1(X_{t_1})\ldots f_n(X_{t_n})\right]\right]. \tag{B.1}$$

For $n=1$, the property holds for any f of the form $f = 1_A$, $A \in \mathcal{B}(\mathbb{R}^d)$ by the definition of a Markov canonical class, see (5.1). Then by pointwise approximation of any positive function $f \in \mathcal{B}_b(\mathbb{R}^d,\mathbb{R}^+)$ by an increasing sequence of simple functions and the monotone convergence theorem for the conditional expectation, the property is also true for any $f \in \mathcal{B}_b(\mathbb{R}^d,\mathbb{R}^+)$. This extends to any $f \in \mathcal{B}_b(\mathbb{R}^d,\mathbb{R})$ by setting $f = f_+ - f_-$.

Let now $n \ge 2$ and assume next that the property (B.1) holds for $n-1$. By the tower property of the conditional expectation as well as (5.1) in Definition 5.1, we get

$$\begin{aligned}\mathbb{E}^{s,x}[1_B f_1(X_{t_1})\ldots f_n(X_{t_n})] &= \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}[f_1(X_{t_1})\ldots f_n(X_{t_n}) \mid \mathcal{F}_{t_{n-1}}]\right] \\ &= \mathbb{E}^{s,x}\left[1_B f_1(X_{t_1})\ldots f_{n-1}(X_{t_{n-1}})\mathbb{E}^{s,x}[f_n(X_{t_n}) \mid \mathcal{F}_{t_{n-1}}]\right] \\ &= \mathbb{E}^{s,x}\left[1_B f_1(X_{t_1})\ldots f_{n-1}(X_{t_{n-1}})\mathbb{E}^{t_{n-1},X_{t_{n-1}}}[f_n(X_{t_n})]\right],\end{aligned}$$

where the latter equality holds because of the first step of the induction. Now the function

$$f : (x_1,\ldots,x_{n-1}) \in (\mathbb{R}^d)^{n-1} \mapsto f_1(x_1)\ldots f_n(x_{n-1})\mathbb{E}^{t_{n-1},x_{n-1}}[f_n(X_{t_n})]$$

belongs to $\mathcal{B}_b((\mathbb{R}^d)^{n-1},\mathbb{R})$. By the tower property and the induction step $n-1$ we get (B.1) for the integer n.

From the linearity and the monotone convergence theorem of the conditional expectation, we see that the class $\mathcal{H} := \{A \in \mathcal{B}(\mathbb{R}^d)^{\otimes n} \mid \mathbb{E}^{s,x}[1_B 1_A] = \mathbb{E}^{s,x}\left[1_B \mathbb{E}^{t,X_t}[1_A]\right]\}$ is a monotone class (λ-system). From (B.1), applied with $f_k = 1_{A_k}$ for some $A_k \in \mathcal{B}(\mathbb{R}^d)$, $1 \le k \le n$, we see that $\mathcal{H}$ contains the π-system $\mathcal{P} := \{A \in \mathcal{B}(\mathbb{R}^d)^{\otimes n} \mid A = A_1 \times \ldots \times A_n, A_1,\ldots,A_n \in \mathcal{B}(\mathbb{R}^d)\}$. Since $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}^d)^{\otimes n}$ it follows from Theorem 19, Chapter I in [16] that $\mathcal{H} = \mathcal{B}(\mathbb{R}^d)^{\otimes n}$. Finally by approximation of any positive function $f \in \mathcal{B}_b((\mathbb{R}^d)^n,\mathbb{R}^+)$ by an increasing sequence of simple functions and the monotone convergence theorem for the conditional expectation, it holds for any $f \in \mathcal{B}_b((\mathbb{R}^d)^n,\mathbb{R}^+)$ that

$$\mathbb{E}^{s,x}\left[1_B f(X_{t_1},\ldots,X_{t_n}))\right] = \mathbb{E}^{s,x}\left[1_B \mathbb{E}^{t,X_t}[f(X_{t_1},\ldots,X_{t_n})]\right], \tag{B.2}$$

and (B.2) can be extended to any $f \in \mathcal{B}_b((\mathbb{R}^d)^n,\mathbb{R})$, by setting $f = f_+ - f_-$. Finally the induction property is verified and the conclusion follows. □

Proof of Proposition B.1. Let

$$\mathcal{H} := \left\{ F \in \mathcal{B}_b(C([t,T]),\mathbb{R}) \mid \mathbb{E}^{s,x}\left[F((X_u)_{u\in[0,T]}) \mid \mathcal{F}_t\right] = \mathbb{E}^{t,X_t}[F((X_u)_{u\in[0,T]})] \right\}.$$

By linearity of the conditional expectation, $\mathcal{H}$ is a vector space. By monotone convergence of the conditional expectation, if $(F_n)_{n\geq 1}$ is a non-negative increasing sequence of elements of $\mathcal{H}$ such that $0 \leq F_n \leq F_{n+1}$ for all $n \geq 1$, then $\sup_{n\geq 1} F_n \in \mathcal{H}$. Finally $\mathcal{C}$ be the class of all cylindrical sets on $D([t,T],\mathbb{R}^d)$, that is

$$\mathcal{C} = \left\{ \{X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\} \mid n \in \mathbb{N}, t \leq t_1 \leq \ldots \leq t_n, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}^d) \right\}.$$

Then we get from Lemma B.2 that $1_C \in \mathcal{H}$ for all $C \in \mathcal{C}$ and by Theorem 21, Chapter I in [16] $\mathcal{H} = \mathcal{B}_b(D[t,T],\mathbb{R})$. □

We generalize slightly Proposition B.1 in the case when F is a non-negative measurable function, not necessarily bounded. We recall to this aim the existence of a generalized version of the conditional expectation for non-negative random variable. We refer to Remark 39, Chapter I in [16].

Proposition B.3. *(Generalized conditional expectation). Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let M be a **non-negative** random variable on $(\Omega,\mathcal{F})$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. There exists a unique non-negative $\mathcal{G}$-measurable random variable with values in $[0,+\infty]$, denoted $\mathbb{E}[Y|\mathcal{G}]$, such that for $\mathbb{E}[1_A Y] = \mathbb{E}[1_A \mathbb{E}[Y|\mathcal{G}]]$ for all $A \in \mathcal{G}$.*

Proposition B.4. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be a Markov canonical class in the sense of Definition 5.1. Let $F \in \mathcal{B}(D([t,T],\mathbb{R}^d),[0,+\infty])$. Then*

$$\mathbb{E}^{s,x}\left[F((X_u)_{u\in[t,T]})\middle|\mathcal{F}_t\right] = \mathbb{E}^{t,X_t}\left[F((X_u)_{u\in[t,T]})\right] \ \mathbb{P}^{s,x}\text{-a.s.} \tag{B.3}$$

Proof. Let $n \in \mathbb{N}$. By Proposition B.1, (B.3) holds for F replaced by $F \wedge n$, and we have

$$\mathbb{E}^{s,x}\left[F((X_u)_{u\in[t,T]}) \wedge n\middle|\mathcal{F}_t\right] = \mathbb{E}^{t,X_t}\left[F((X_u)_{u\in[t,T]}) \wedge n\right] \ \mathbb{P}^{s,x}\text{-a.s.} \tag{B.4}$$

On the one hand, by the monotone convergence theorem for the conditional expectation, $\mathbb{P}^{s,x}$-a.s. we have

$$\mathbb{E}^{s,x}\left[F((X_u)_{u\in[t,T]}) \wedge n\middle|\mathcal{F}_t\right] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}^{s,x}\left[F((X_u)_{u\in[t,T]})\middle|\mathcal{F}_t\right]. \tag{B.5}$$

On the other hand, for all $y \in \mathbb{R}^d$, by monotone convergence

$$\mathbb{E}^{t,y}\left[F((X_u)_{u\in[t,T]}) \wedge n\right] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}^{t,y}\left[F((X_u)_{u\in[t,T]})\right],$$

hence

$$\mathbb{E}^{t,X_t}\left[F((X_u)_{u\in[t,T]}) \wedge n\right] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}^{t,X_t}\left[F((X_u)_{u\in[t,T]})\right] \ \mathbb{P}^{s,x}\text{-a.s.} \tag{B.6}$$

We emphasize that the conditional expectation in the right-hand side of (B.5) and (B.6) are to be understood in the sense of Proposition B.3. This shows the validity of (B.3). □

Lemma B.5. *Let $\mathbb{P} \in \mathcal{P}(\Omega)$ be a Regularly Markovian probability measure in the sense of Definition 5.7 with associated Markov canonical class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ and initial law μ. Then for all $t \in [0,T]$, $F \in \mathcal{B}_b(D([t,T],\mathbb{R}^d),\mathbb{R})$,*

$$\mathbb{E}^{\mathbb{P}}[F((X_u)_{u\in[t,T]})|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[F((X_u)_{u\in[t,T]})|X_t] = \mathbb{E}^{t,X_t}[F((X_u)_{u\in[t,T]})] \quad \mathbb{P}\text{-a.s.}$$

In particular $\mathbb{P}$ verifies the Markov Property Hypothesis 3.3.

Proof. We set $Z := F((X_u)_{u\in[t,T]})$. Let $A \in \mathcal{F}_t$. By definition of $\mathbb{P}$ as well as by Proposition B.1 we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[Z1_A] &= \int_{\mathbb{R}^d} \mathbb{E}^{0,x}[Z1_A]\mu(dx)\\ &= \int_{\mathbb{R}^d} \mathbb{E}^{0,x}\left[\mathbb{E}^{0,x}[Z|\mathcal{F}_t]1_A\right]\mu(dx)\\ &= \int_{\mathbb{R}^d} \mathbb{E}^{0,x}[\mathbb{E}^{t,X_t}[Z]1_A]\mu(dx)\\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{t,X_t}[Z]1_A\right]\end{aligned}$$

and the conclusion follows immediately from the last equality by the definition of the conditional expectation. □

Lemma B.6. *Let $\mathbb{P}$ our reference probability supposed to fulfill Markov property Hypothesis 3.3.*
Then, the probability $\mathbb{Q}^*$ defined by (3.8)*. also verifies the same Markov property.*

Proof. Let $t \in [0,T]$ and $F \in \mathcal{B}(D([t,T],\mathbb{R}^d),\mathbb{R})$. It holds

$$\mathbb{E}^{\mathbb{Q}^*}\left[F\left((X_r)_{r\in[t,T]}\right)\Big|\mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[F\left((X_r)_{r\in[t,T]}\right)\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|\mathcal{F}_t\right]}. \tag{B.7}$$

Then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_0^T f(r,X_r)dr - g(X_T)\right)\Big|\mathcal{F}_t\right]\\ &= \exp\left(-\int_0^t f(r,X_r)dr\right)\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\Big|\mathcal{F}_t\right]\\ &= \exp\left(-\int_0^t f(r,X_r)dr\right)\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\Big|X_t\right],\end{aligned} \tag{B.8}$$

where we have used Lemma B.5 for the latter equality. Similarly,

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[F\left((X_r)_{r\in[t,T]}\right)\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|\mathcal{F}_t\right]\\ &= \exp\left(-\int_0^t f(r,X_r)dr\right)\mathbb{E}^{\mathbb{P}}\left[F\left((X_r)_{r\in[t,T]}\right)\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\Big|X_t\right].\end{aligned} \tag{B.9}$$

Combining (B.8) and (B.9) with (B.7), we get

$$\mathbb{E}^{\mathbb{Q}^*}\left[F\left((X_r)_{r\in[t,T]}\right)\Big|\mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[F\left((X_r)_{r\in[t,T]}\right)\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\Big|X_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T f(r,X_r)dr - g(X_T)\right)\Big|X_t\right]}. \tag{B.10}$$

This concludes the proof. □

Appendix C: Proof of Proposition 5.11

(a) We first prove item 1. The existence and uniqueness of $\Gamma(\phi,\psi)$ for any fixed ϕ and ψ follows directly from Corollary D.12 item 2. It remains to check the bilinearity of the operator Γ. Recall that $\mathcal{D}^2(\mathfrak{a})$ and $\mathcal{D}^2_{dt}(\mathfrak{a})$ are vector spaces, see Remark 5.6. Let then $\lambda, \mu \in \mathbb{R}$, $\phi_1, \phi_2 \in \mathcal{D}^2_{dt}(\mathfrak{a})$, $\psi \in \mathcal{D}^2(\mathfrak{a})$. Let $(s,x) \in [0,T] \times \mathbb{R}^d$. It holds that $M[\lambda\phi_1 + \mu\phi_2]^{s,x} = \lambda M[\phi_1]^{s,x} + \mu M[\phi_2]^{s,x}$ $\mathbb{P}^{s,x}$-a.s. and

$$\langle M[\lambda\phi_1 + \mu\phi_2]^{s,x}, M[\psi]^{s,x}\rangle_\cdot = \lambda\langle M[\phi_1]^{s,x}, M[\psi]^{s,x}\rangle_\cdot + \mu\langle M[\phi_2]^{s,x}, M[\psi]^{s,x}\rangle_\cdot \quad \mathbb{P}^{s,x}\text{-a.s.},$$

which yields

$$\langle M[\lambda\phi_1 + \mu\phi_2]^{s,x}, M[\psi]^{s,x}\rangle_\cdot = 1_{[s,T]}(\cdot)\int_s^\cdot \Gamma(\lambda\phi_1 + \mu\phi_2, \psi)(r, X_r)dr \quad \mathbb{P}^{s,x}\text{-a.s.}$$

and

$$\langle M[\lambda\phi_1 + \mu\phi_2]^{s,x}, M[\psi]^{s,x}\rangle_\cdot = \lambda 1_{[s,T]}(\cdot)\int_s^\cdot \Gamma(\phi_1, \psi)(r, X_r)dr + \mu 1_{[s,T]}(\cdot)\int_s^\cdot \Gamma(\phi_2, \psi)(r, X_r)dr \quad \mathbb{P}^{s,x}\text{-a.s.}$$

The previous equalities hold true for all $(s,x) \in [0,T] \times \mathbb{R}^d$ and by uniqueness of $\Gamma(\lambda\phi_1 + \mu\phi_2, \psi)$ up to a zero-potential set, we get $\Gamma(\lambda\phi_1 + \mu\phi_2, \psi) = \lambda\Gamma(\phi_1, \psi) + \mu\Gamma(\phi_2, \psi)$. Similarly for all $\phi \in \mathcal{D}^2_{dt}(\mathfrak{a})$, $\psi_1, \psi_2 \in \mathcal{D}^2(\mathfrak{a})$, we get $\Gamma(\phi, \lambda\psi_1 + \mu\psi_2) = \lambda\Gamma(\phi, \psi_1) + \mu\Gamma(\phi, \psi_2)$, hence the linearity of Γ in the first variable ϕ. The linearity in the second variable ψ follows by almost identical arguments. This proves item 1.

(b) Let $\phi \in \mathcal{D}^2_{dt}(\mathfrak{a})$. We first show that

$$M[\phi] \in \mathcal{H}^2_{loc}(\mathbb{P}). \tag{C.1}$$

Let $(\tau_n)_{n\geq 1}$ be the sequence of stopping times given by

$$\tau_n := \inf\left\{t \in [0,T] \ : \ \int_0^t \Gamma(\phi,\phi)(r, X_r)dr > n\right\},$$

where $\Gamma(\phi,\phi)$ is given by item 1. which fulfills (5.5). Then for any $x \in \mathbb{R}^d$,

$$\mathbb{E}^{0,x}\left[[M[\phi]^{0,x}]_{T\wedge\tau_n}\right] = \mathbb{E}^{0,x}\left[\langle M[\phi]^{0,x}\rangle_{T\wedge\tau_n}\right] = \mathbb{E}^{0,x}\left[\int_0^{T\wedge\tau_n}\Gamma(\phi,\phi)(r, X_r)dr\right] \leq n$$

and $M[\phi]^{0,x}_{\cdot\wedge\tau_n}$ is an element of $\mathcal{H}^2(\mathbb{P}^{s,x})$. Furthermore, for all $t \in [0,T]$,

$$\mathbb{E}^{\mathbb{P}}[|M[\phi]_t|^2] = \int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[|M[\phi]_t|^2\right]\mu(dx) = \int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[|M[\phi]^{0,x}_t|^2\right]\mu(dx) \leq n.$$

and $M[\phi]_t \in L^2(\mathbb{P})$. Let then $0 \leq t < u \leq T$ be fixed and $F \in \mathcal{F}_t$. Since $M[\phi]^{0,x}_{\cdot\wedge\tau_n}$ is a $\mathbb{P}^{0,x}$-martingale for all $x \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[1_F M[\phi]_{u\wedge\tau_n}\right] &= \int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F M[\phi]^{0,x}_{u\wedge\tau_n}\right]\mu(dx) = \int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F M[\phi]^{0,x}_{t\wedge\tau_n}\right]\mu(dx)\\ &= \mathbb{E}^{\mathbb{P}}\left[1_F M[\phi]_{t\wedge\tau_n}\right].\end{aligned}$$

It follows from what precedes that $M[\phi]_{\cdot\wedge\tau_n} \in \mathcal{H}^2(\mathbb{P})$, that is (C.1).

(c) Before proving item 2. we make some preliminary computations. We recall that $v \in \mathcal{D}^2(\mathfrak{a})$ by Remark 5.10. Set $\Gamma^v(\phi) := \Gamma(\phi, v)$ for all $\phi \in \mathcal{D}^2_{dt}(\mathfrak{a})$, where Γ is given by item 1. Clearly Γ^v is linear since Γ is bilinear, again by item 1. Assume for a moment that there exists a localizing sequence $(\mathcal{S}_n)_{n\geq 1}$ such that $(M[\phi]M[v])_{t\wedge\mathcal{S}_n} - \int_0^{t\wedge\mathcal{S}_n}\Gamma^v(\phi)(r, X_r)dr$ belongs to $L^1(\mathbb{P})$ for all $t \in [0,T]$ and $(M[\phi]^{0,x}M[v]^{0,x})_{\cdot\wedge\mathcal{S}_n} - \langle M[\phi]^{0,x}, M[v]^{0,x}\rangle_{\cdot\wedge\mathcal{S}_n}$ is a genuine $\mathbb{P}^{0,x}$-martingale for μ-almost all $x \in \mathbb{R}^d$. We will show below that

$$\langle M[\phi], M[v]\rangle_\cdot = \int_0^\cdot \Gamma^v(\phi)(r, X_r)dr \quad \mathbb{P}\text{-a.s.} \tag{C.2}$$

Indeed, for all $0 \le t < u \le T$ and $F \in \mathcal{F}_t$, (5.5) implies that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[1_F\left((M[\phi]M[v])_{u\wedge\mathcal{S}_n} - \int_0^{u\wedge\mathcal{S}_n}\Gamma^v(\phi)(r,X_r)dr\right)\right]\\
&=\int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F\left((M[\phi]^{0,x}M[v]^{0,x})_{u\wedge\mathcal{S}_n} - \int_0^{u\wedge\mathcal{S}_n}\Gamma^v(\phi)(r,X_r)dr\right)\right]\mu(dx)\\
&=\int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F\left((M[\phi]^{0,x}M[v]^{0,x})_{u\wedge\mathcal{S}_n} - \langle M[\phi]^{0,x}, M[v]^{0,x}\rangle_{u\wedge\mathcal{S}_n}\right)\right]\mu(dx)\\
&=\int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F\left((M[\phi]^{0,x}M[v]^{0,x})_{t\wedge\mathcal{S}_n} - \langle M[\phi]^{0,x}, M[v]^{0,x}\rangle_{t\wedge\mathcal{S}_n}\right)\right]\mu(dx)\\
&=\int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F\left((M[\phi]^{0,x}M[v]^{0,x})_{t\wedge\mathcal{S}_n} - \int_0^{t\wedge\mathcal{S}_n}\Gamma^v(\phi)(r,X_r)dr\right)\right]\mu(dx)\\
&=\mathbb{E}^{\mathbb{P}}\left[1_F\left((M[\phi]M[v])_{t\wedge\mathcal{S}_n} - \int_0^{t\wedge\mathcal{S}_n}\Gamma^v(\phi)(r,X_r)dr\right)\right].
\end{aligned}
\tag{C.3}
$$

It follows that the process $(M[\phi]M[v])_{\cdot\wedge\mathcal{S}_n} - \int_0^{\cdot\wedge\mathcal{S}_n}\Gamma^v(\phi)(r,X_r)dr$ would be a $(\mathbb{P},\mathcal{F}_t)$-martingale. Hence $(M[\phi]M[v])_\cdot - \int_0^\cdot \Gamma^v(\phi)(r,X_r)dr$ would be a $(\mathbb{P},\mathcal{F}_t)$-local martingale and since the process $\int_0^\cdot \Gamma^v(\phi)(r,X_r)dr$ is predictable, we could conclude (C.2).

(d) We prove now item 2. Let $(\sigma_n)_{n\ge 1}$ be the sequence of stopping times given by

$$
\sigma_n := \inf\left\{t\in[0,T]\ :\ \int_0^t f(r,X_r)v(r,X_r)dr \ge n\right\}.
$$

As v is bounded, the process $M[v]^{0,x}_{\cdot\wedge\mathcal{S}_n}$ is bounded by $L_n := 2\|v\|_\infty + n$ for all $x\in\mathbb{R}^d$. Moreover, since $M[v]^{0,x}_{\cdot\wedge\sigma_n}$ is a bounded local martingale, it is a genuine $\mathbb{P}^{0,x}$-martingale and we deduce that $(\sigma_n)_{n\ge1}$ is a localizing sequence for the local martingale $M[v]^{0,x}$ for all $x\in\mathbb{R}^d$. Hence for any $0\le t\le u\le T$, $F\in\mathcal{F}_t$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[1_F M[v]_{u\wedge\sigma_n}\right] &= \int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F M[v]^{0,x}_{u\wedge\sigma_n}\right]\mu(dx) = \int_{\mathbb{R}^d}\mathbb{E}^{0,x}\left[1_F M[v]^{0,x}_{t\wedge\sigma_n}\right]\mu(dx)\\
&= \mathbb{E}^{\mathbb{P}}\left[1_F M[v]_{t\wedge\sigma_n}\right].
\end{aligned}
$$

The process $M[v]$ is a locally bounded $\mathbb{P}$-local martingale, hence $M[v]\in\mathcal{H}^2_{loc}(\mathbb{P})$. Let $\phi\in\mathcal{D}^2_{dt}(\mathfrak{a})$. We set

$$
\mathcal{S}_n := \inf\left\{t\in[0,T]\ :\ \int_0^t \Gamma(\phi,\phi)(r,X_r)dr \ge n \text{ or } \int_0^t |\Gamma^v(\phi)|(r,X_r)dr\ge n\right\}\wedge\sigma_n.
$$

The process $N := (M[\phi]^{0,x}M[v]^{0,x})_{\cdot\wedge\mathcal{S}_n} - \langle M[\phi]^{0,x}, M[v]^{0,x}\rangle_{\cdot\wedge\mathcal{S}_n}$ is a $\mathbb{P}^{0,x}$-local martingale and for μ-almost all $x \in \mathbb{R}^d$, for all $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}^{0,x}\left[\sup_{0\le t\le T}|N_t|\right] &\le \mathbb{E}^{0,x}\left[\sup_{0\le t\le T}\left|(M[\phi]^{0,x}M[v]^{0,x})_{t\wedge\mathcal{S}_n}\right|\right] + \mathbb{E}^{0,x}\left[\sup_{0\le t\le T}\left|\langle M[\phi]^{0,x}, M[v]^{0,x}\rangle_{t\wedge\mathcal{S}_n}\right|\right]\\
&\le L_n\mathbb{E}^{0,x}\left[\sup_{0\le t\le T}(M[\phi]^{0,x}_{t\wedge\mathcal{S}_n})^2\right]^{1/2} + \mathbb{E}^{0,x}\left[\int_0^{T\wedge\mathcal{S}_n}|\Gamma^v(\phi)|(r,X_r)dr\right]\\
&\le 2L_n\mathbb{E}^{0,x}\left[\langle M[\phi]^{0,x}\rangle_{T\wedge\mathcal{S}_n}\right]^{1/2} + \mathbb{E}^{0,x}\left[\int_0^{T\wedge\mathcal{S}_n}|\Gamma^v(\phi)|(r,X_r)dr\right]\\
&\le 2L_n\mathbb{E}^{0,x}\left[\int_0^{T\wedge\mathcal{S}_n}\Gamma(\phi,\phi)(r,X_r)dr\right]^{1/2} + n\\
&\le 2L_n\sqrt{n} + n,
\end{aligned}$$

where we have used the BDG inequality for càdlàg local martingales (see e.g. [32]), recalling that by definition of $\mathcal{S}_n$ and (5.5), $\mathbb{E}^{0,x}\left[\langle M[\phi]^{0,x}\rangle_{t\wedge\mathcal{S}_n}\right] = \mathbb{E}^{0,x}\left[\int_0^{t\wedge\mathcal{S}_n}\Gamma(\phi,\phi)(r,X_r)dr\right] \le n$. It follows that $(M[\phi]^{0,x}M[v]^{0,x})_{\cdot\wedge\mathcal{S}_n} - \langle M[\phi]^{0,x}, M[v]^{0,x}\rangle_{\cdot\wedge\mathcal{S}_n}$ is a genuine $\mathbb{P}^{0,x}$-martingale all $x \in \mathbb{R}^d$. This realizes the program of item (c) which implies (C.2) and finally (5.6) is verified, hence item 2.

Appendix D: Local Martingale Additive Functionals

The aim of this section is to extend the results of [4] to the case of (what we call) locally square integrable Local Martingale Additive Functionals. We emphasize that the following is just a careful reading of the proofs of the results in [4] with slight modifications, and that all the technicalities have already been treated by the authors of [4]. In the sequel we will denote $\Delta := \{(t,u) \in [0,T] \ : \ t \le u\}$. In the whole section $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ will be a measurable in time Markov canonical class in the sense of Definition 5.1. For the sake of conciseness, the $\mathbb{P}^{s,x}$-completion of any sub-σ-field $\mathcal{G} \subset \mathcal{F}$ will be denoted $\mathcal{G}^{s,x}$ instead of $\mathcal{G}^{\mathbb{P}^{s,x}}$. The following is Definition 4.1 in [4].

Definition D.1. *On $(\Omega, \mathcal{F})$ we define a (non-homogeneous) Additive Functional (AF) as a random-field $A := (A^t_u)_{(t,u)\in\Delta}$ indexed by Δ with values in $\mathbb{R}$ verifying the two following conditions.*

1. *For any $(t,u) \in \Delta$, A^t_u is $\mathcal{F}_{t,u}$-measurable, where those σ-fields were defined in Section 2*
2. *For any $(s,x) \in [0,T] \times \mathbb{R}^d$ there exists real càdlàg $(\mathcal{F}^{s,x}_t)$-adapted process $A^{s,x}$ (taken equal to 0 on $[0,s[$ by convention) such that for any $x \in \mathbb{R}^d$ and $s \le t \le u$, $A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$-a.s. for every (s,x).*

$A^{s,x}$ will be called the càdlàg version of A under $\mathbb{P}^{s,x}$.

Definition D.2. *(Local Martingale Additive Functional). Let $A := (A^t_u)_{(t,u)\in\Delta}$ be an AF in the sense of Definition D.1. Let $A^{s,x}$ be its càdlàg version under $\mathbb{P}^{s,x}$. A will be called a Local Martingale Additive Functional (LMAF) (resp. a locally square integrable LMAF) if $A^{s,x}$ is a local martingale (resp. a locally square integrable local martingale).*

More generally, an AF A will be said to verify a certain property (being increasing, non-negative, locally integrable...) if its càdlàg version $A^{s,x}$ satisfies this property under $\mathbb{P}^{s,x}$.

We start by a small lemma of independent interest which is more or less the end of the proof of Proposition 4.4 in [4].

Lemma D.3. *Let $t \in]0,T[$ be fixed. Let $\mathcal{G} \subset \mathcal{F}_{t,T}$ be a sub-σ-algebra of $\mathcal{F}_{t,T}$. Let $(M_n)_{n\ge1}$ be a sequence of $\mathcal{G}$-measurable random variables such that for all $(s,x) \in [0,t] \times \mathbb{R}^d$ the sequence $(M_n)_{n\ge1}$ converges in probability under $\mathbb{P}^{s,x}$. Then there exists a $\mathcal{G}$-measurable random variable M such that*

$$M_n \xrightarrow[n\to+\infty]{\mathbb{P}^{s,x}} M, \ \forall (s,x) \in [0,t] \times \mathbb{R}^d. \tag{D.1}$$

Proof. The difficulty comes from the fact that the convergences in the statement are in probability $\mathbb{P}^{t,x}$ and not $\mathbb{P}^{t,x}$-a.s. We first assume that for all $n \geq 1$, $M_n \geq 0$. For all $x \in \mathbb{R}^d$, denote $M^{t,x}$ the limit of $(M_n)_{n\geq 1}$ under $\mathbb{P}^{t,x}$, namely

(D.2) $$M_n \underset{n\to+\infty}{\overset{\mathbb{P}^{t,x}}{\longrightarrow}} M^{t,x}.$$

Since M_n is $\mathcal{G}$-measurable for all $n \geq 1$, $M^{t,x}$ is $\mathcal{G}^{t,x}$-measurable and by Proposition 3.12 in [4], there exists a $\mathcal{G}$-measurable random variable $a_t(x,\omega)$ such that

(D.3) $$a_t(x,\omega) = M^{t,x} \ \mathbb{P}^{t,x}\text{-a.s.}$$

Let us prove that there exists a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$-measurable version of $(x,\omega) \mapsto a_t(x,\omega)$. For any integer $k \geq 1$, we set $a^k_t(x,\omega) := k \wedge a_t(x,\omega)$. We then define for all $x \in \mathbb{R}^d$, $k \geq 1$ and $n \geq 1$ two positive measures $\mathbb{Q}^{x,k,n}$ and $\mathbb{Q}^{x,k}$ on $\mathcal{G}$ as follows, for all $G \in \mathcal{G}$:

i) $\mathbb{Q}^{x,k,n}(G) := \mathbb{E}^{t,x}[1_G(k \wedge M_n)]$;
ii) $\mathbb{Q}^{x,k}(G) := \mathbb{E}^{t,x}[1_G a^k_t(x,\omega)]$.

It is clear that $k \wedge M_n$ converges in probability towards $k \wedge M^{t,x}$ under $\mathbb{P}^{t,x}$. Since the sequence $(k \wedge M_n)_{n\geq 1}$ is uniformly bounded by k, the convergence also takes place in $L^1(\mathbb{P}^{t,x})$. Hence, for any $G \in \mathcal{G}$,

$$x\mathbb{Q}^{x,k,n}(G) \underset{n\to+\infty}{\longrightarrow} \mathbb{E}^{t,x}[1_G(k \wedge M^{t,x})] = \mathbb{E}^{t,x}[1_G a^k_t(x,\omega)] = \mathbb{Q}^{x,k}(G),$$

where we used (D.3) for the first equality. Moreover, for fixed $G \in \mathcal{G}$, by Remark 5.3, $x \mapsto \mathbb{Q}^{x,k,n}(G)$ is Borel. Thus $x \mapsto \mathbb{Q}^{x,k}(G)$ is Borel being a pointwise limit of Borel functions. We recall that $\mathcal{F}$ is separable. Combining these two previous properties, along with the fact that $\mathbb{Q}^{x,k} \ll \mathbb{P}^{t,x}$ allows to prove the existence of a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$-measurable version of the density $d\mathbb{Q}^{x,k}/d\mathbb{P}^{t,x}$. This fact follows from Theorem 58, Chapter V in [17]. Hence there exists a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$-measurable version of $a^k_t(x,\omega)$, still denoted $a^k_t(x,\omega)$. Then $a_t(x,\omega) = \liminf\limits_{k\to+\infty} a^k_t(x,\omega)$ is also $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$-measurable. We can now set $M(\omega) := a_t(X_t(\omega),\omega)$. Clearly, for any $x \in \mathbb{R}^d$ it holds by (D.3) that

(D.4) $$M^{t,x} = M \ \mathbb{P}^{t,x}\text{-a.s.}$$

It remains to prove (D.1). Let $(s,x) \in [0,t] \times \mathbb{R}^d$. Let $\epsilon > 0$. Recall that $M_n - M$ is $\mathcal{G}$-measurable and that $\mathcal{G} \subset \mathcal{F}_{t,T}$. We have

$$\begin{aligned}\mathbb{P}^{s,x}(|M_n - M| > \epsilon) &= \mathbb{E}^{s,x}\left[\mathbb{P}^{s,x}(|M_n - M| > \epsilon | X_t)\right] && \text{(tower property)}\\ &= \mathbb{E}^{s,x}[\mathbb{P}^{t,X_t}(|M_n - M| > \epsilon)] && \text{(Markov property (3.3)).}\end{aligned}$$

For any fixed $\omega \in \Omega$, $\mathbb{P}^{t,X_t(\omega)}(|M_n - M| > \epsilon) \to 0$ as $n \to +\infty$ by (D.2) and (D.4), hence by dominated convergence theorem, $\mathbb{E}^{s,x}[\mathbb{P}^{t,X_t}(|M_n - M| > \epsilon)] \underset{n\to+\infty}{\longrightarrow} 0$, which implies (D.1). Now if M_n are not non-negative, setting $M_n^+ := (M_n)_+$ and $M_n^- := (M_n)_-$ and applying what precedes, we get the existence of two $\mathcal{G}$-measurable random variable M^+ and M^- satisfying (D.1). Let $M := M^+ - M^-$. Then M is $\mathcal{G}$-measurable and (D.1) is verified. This concludes the proof. □

We will often make use of Lemma D.3 in the sequel of this section. The following result generalizes Proposition 4.4 in [4].

Proposition D.4. *Let $(M^t_u)_{(t,u)\in\Delta}$ be a LMAF and for any $(s,x) \in [0,T] \times \mathbb{R}^d$, $[M^{s,x}]$ be the quadratic variation of its càdlàg version $M^{s,x}$ under $\mathbb{P}^{s,x}$. Then there exists an AF denoted $([M]^t_u)_{(t,u)\in\Delta}$ and which has càdlàg version $[M^{s,x}]$ under $\mathbb{P}^{s,x}$ for any $(s,x) \in [0,T] \times \mathbb{R}^d$.*

Proof. Let $0 \leq t < u \leq T$ be fixed. Let $t = t_1^n < t_2^n < \ldots < t_n^n = u$ be a sequence of subdivisions of the interval $[t, u]$ such that $\max_{i<n}(t_{i+1}^n - t_i^n) \underset{n\to+\infty}{\longrightarrow} 0$. By definition of the quadratic variation $\mathbb{P}^{s,x}$-a.s. it holds that

$$\sum_{i<n}\left(M_{t_{i+1}^n}^{t_i^n}\right)^2 = \sum_{i<n}\left(M_{t_{i+1}^n}^{s,x} - M_{t_i^n}^{s,x}\right)^2 \underset{n\to+\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} [M^{s,x}]_u - [M^{s,x}]_t, \tag{D.5}$$

for all $(s,x) \in [0,t] \times \mathbb{R}^d$. The random variable $\sum_{i<n}\left(M_{t_{i+1}^n}^{t_i^n}\right)^2$ is $\mathcal{F}_{t,u}$-measurable for all $n \geq 1$ and $\mathcal{F}_{t,u} \subset \mathcal{F}_{t,T}$. Lemma D.3 applied with $M_n = \sum_{i<n}\left(M_{t_{i+1}^n}^{t_i^n}\right)^2$ and $\mathcal{G} = \mathcal{F}_{t,u}$ gives the existence of an $\mathcal{F}_{t,u}$-random variable M_t^u such that

$$\sum_{i<n}\left(M_{t_{i+1}^n}^{t_i^n}\right)^2 \underset{n\to+\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} M_t^u, \tag{D.6}$$

for all $(s,x) \in [0,t] \times \mathbb{R}^d$. The identification of the limits (D.5) and (D.6) yields $M_t^u = [M^{s,x}]_u - [M^{s,x}]_t$ $\mathbb{P}^{s,x}$-a.s. for all $(s,x) \in [0,t] \times \mathbb{R}^d$. □

In the following me make use of the notions of *predictable projection*, see Theorem 2.28, Chapter I in [26]. We will also make use of the notion of *compensator* of a bounded variation process (resp. random measure) introduced in Theorem 3.18, Chapter I (resp. Theorem 1.8, Chapter II) in [26].

Proposition D.5. *Let $(B_u^t)_{(t,u)\in\Delta}$ be a locally integrable increasing AF with corresponding càdlàg version $B^{s,x}$ under $\mathbb{P}^{s,x}$ for $(s,x) \in [0,T] \times \mathbb{R}^d$. Let $A^{s,x}$ be the compensator of $B^{s,x}$ in $(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x})$. Then there exists a locally integrable increasing AF $A := (A_u^t)_{(t,u)\in\Delta}$ so that, under any $\mathbb{P}^{s,x}$, the càdlàg version of A is $A^{s,x}$.*

Proof. Let $0 \leq t < u \leq T$ be fixed. The proof follows the same lines as Proposition 4.5 in [4]. In that paper $(B_u^t)_{(t,u)\in\Delta}$ has L^1-terminal value. We split the proof in two steps. The first one consists in showing that

$$\forall (s,x) \in [0,t] \times \mathbb{R}^d,\ A_u^{s,x} - A_t^{s,x} \text{ is } \mathcal{F}_{t,u}^{s,x}\text{-measurable.} \tag{D.7}$$

The second part of the proof consists in showing the existence of an $\mathcal{F}_{t,u}$-measurable random variable A_u^t satisfying $A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathbb{P}^{s,x}$-a.s. for all $(s,x) \in [0,t] \times \mathbb{R}^d$.

1. Let us fix $(s,x) \in [0,T] \times \mathbb{R}^d$. For any $F \in \mathcal{F}^{s,x}$, let $N^{s,x,F}$ be the càdlàg version of the martingale $r \mapsto \mathbb{E}^{s,x}[1_F|\mathcal{F}_r]$. Then the predictable projection of the process $r \mapsto 1_F 1_{[t,u[}(r)$ is $r \mapsto N_{r-}^{s,x,F} 1_{[t,u[}(r)$, which is a non-negative bounded predictable process. Let $(\tau_n)_{n\geq1}$ be a localizing sequence for the process $B^{s,x}$. Then $A_{\cdot\wedge\tau_n}^{s,x}$ is the compensator of $B_{\cdot\wedge\tau_n}^{s,x}$, see for example 3.20, Chapter I in [26], and by Definition 73, Chapter VI in [17] it holds that

$$\mathbb{E}^{s,x}\left[1_F\left(A_{u\wedge\tau_n}^{s,x} - A_{t\wedge\tau_n}^{s,x}\right)\right] = \mathbb{E}^{s,x}\left[\int_{t\wedge\tau_n}^{u\wedge\tau_n} N_{r-}^{s,x,F} dB_r^{s,x}\right]. \tag{D.8}$$

The sequences $\left(A_{u\wedge\tau_n}^{s,x} - A_{t\wedge\tau_n}^{s,x}\right)_{n\geq1}$ and $\left(\int_{t\wedge\tau_n}^{u\wedge\tau_n} N_{r-}^{s,x,F} dB_r^{s,x}\right)_{n\geq1}$ are $\mathbb{P}^{s,x}$-a.s. increasing after a certain rank. Letting $n \to +\infty$ in (D.8) yields by monotone convergence

$$\mathbb{E}^{s,x}\left[1_F\left(A_u^{s,x} - A_t^{s,x}\right)\right] = \mathbb{E}^{s,x}\left[\int_t^u N_{r-}^{s,x,F} dB_r^{s,x}\right]. \tag{D.9}$$

Assume now that $F \in \mathcal{F}_{t,T}$. Lemma 4.7 in [4] provides the existence of an $\mathcal{F}_{t,u}$-random variable denoted $\int_t^u N_{r-}^F dB_r$ such that for all $(s,x) \in [0,t] \times \mathbb{R}^d$, $\int_t^u N_{r-}^F dB_r = \int_t^u N_{r-}^{s,x,F} dB_r^{s,x}$ $\mathbb{P}^{s,x}$-a.s., and in particular it holds

$$\mathbb{E}^{s,x}\left[1_F\left(A_u^{s,x} - A_t^{s,x}\right)\right] = \mathbb{E}^{s,x}\left[\int_t^u N_{r-}^F dB_r\right] \text{ for all } (s,x) \in [0,t] \times \mathbb{R}^d. \tag{D.10}$$

This statement corresponds to the one of Lemma 4.9 in [4]. We prove now

(D.11) $$\mathbb{E}^{s,x}\left[A_u^{s,x} - A_t^{s,x} | \mathcal{F}_{t,T}\right] = A_u^{s,x} - A_t^{s,x} \quad \mathbb{P}^{s,x}\text{-a.s.}$$

For this, the rest of the proof of the first part of Proposition 4.5 in [4] can be reproduced, the only difference being that one has to use the generalized conditional expectation for non-negative random variables, see Proposition B.3. It is necessary in our setting as for instance, the quantity $\mathbb{E}^{s,x}\left[1_F\left(A_u^{s,x} - A_t^{s,x}\right)\right]$ might not be finite. Yet this version of the conditional expectation has the same characterization as the usual conditional expectation and the Markov property still holds, see Proposition B.3 and Proposition B.4. We thus conclude that (D.11) is verified. Finally, by definition, the compensator is adapted so $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_u^{s,x}$-measurable. The relation (D.11) implies that $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,T}^{s,x}$-measurable. Since $\mathcal{F}_{t,u}^{s,x} = \mathcal{F}_u^{s,x} \cap \mathcal{F}_{t,T}^{s,x}$, we deduce that $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,u}^{s,x}$-measurable for all $(s,x) \in [0,t] \times \mathbb{R}^d$, namely (D.7). This concludes the first part of the proof.

2. The second part of the proof follows the proof of the second half of Proposition 4.5 and is very similar to the one of Lemma D.3. From item 1., since $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,u}^{t,x}$-measurable, by Proposition 3.12 in [4] there exists an $\mathcal{F}_{t,u}$-measurable random variable $a_t(x,\omega)$ such that

(D.12) $$a_t(x,\omega) = A_u^{t,x} - A_t^{t,x} \quad \mathbb{P}^{t,x}\text{-a.s.}$$

Let us prove the existence of a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_{t,u}$-measurable version of $(x,\omega) \mapsto a_t(x,\omega)$. For every $x \in \mathbb{R}^d$, we define the positive measure on $\mathcal{F}_{t,u}$

(D.13) $$\mathbb{Q}^x(F) := \mathbb{E}^{t,x}\left[1_F\left(A_u^{t,x} - A_t^{t,x}\right)\right] = \mathbb{E}^{t,x}[1_F a_t(x,\omega)].$$

The measure $\mathbb{Q}^x$ is σ-finite for all $x \in \mathbb{R}^d$. Indeed, we set $E_n := \{\omega \in \Omega : |a_t(x,\omega)| \le n\} \in \mathcal{F}_{t,u}$ for all $n \in \mathbb{N}$, and $E_\infty = \{\omega \in \Omega : |a_t(x,\omega)| = +\infty\} \in \mathcal{F}_{t,u}$. Then $\Omega = \left(\bigcup_{n\in\mathbb{N}} E_n\right) \cup E_\infty$, $\mathbb{Q}^x(E_n) \le n < +\infty$ for all $n \in \mathbb{N}$ and $\mathbb{Q}^x(E_\infty) = 0$ by (D.12), noticing that $A_u^{t,x} - A_t^{t,x}$ is finite $\mathbb{P}^{t,x}$-a.s. Recall that by (D.10) we have

$$\mathbb{Q}^x(F) = \mathbb{E}^{t,x}\left[\int_t^u N_{r-}^F dB_r\right],$$

yielding that $x \mapsto \mathbb{Q}^x(F)$ is Borel for any $F \in \mathcal{F}_{t,u}$ by Remark 5.3. As $\mathbb{Q}^x \ll \mathbb{P}^{t,x}$ and $\mathbb{Q}^x$ is σ-finite for all $x \in \mathbb{R}^d$, Theorem 4 in [15] gives the existence of a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_{t,u}$-measurable version of the density $d\mathbb{Q}^x/d\mathbb{P}^{t,x}$, hence the existence of a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_{t,u}$-measurable version of $(x,\omega) \mapsto a_t(x,\omega)$, still denoted $a_t(x,\omega)$. We now set $A_u^t := a_t(X_t(\omega),\omega)$, which clearly is $\mathcal{F}_{t,u}$-measurable and satisfies $A_u^t = A_u^{t,x} - A_t^{t,x}$ $\mathbb{P}^{t,x}$-a.s. It remains to prove that the previous equality holds for all $0 \le s \le t$. This is a consequence of the Markov property, see the end of the proof of Proposition 4.5 in [4].

Summing up, for all $0 \le t \le u \le T$, there exists an $\mathcal{F}_{t,u}$-measurable random A_u^t such that for all $(s,x) \in [0,t] \times \mathbb{R}^d$, $A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathbb{P}^{s,x}$-a.s. $(A_u^t)_{(t,u)\in\Delta}$ is then the desired AF, which concludes the proof. □

The following result is a direct consequence of Proposition D.5 and the polarization identity.

Corollary D.6. *Let M, N be two locally square integrable LMAFs with respective càdlàg version $M^{s,x}$ and $N^{s,x}$ under $\mathbb{P}^{s,x}$. Then there exists a bounded variation AF denoted $(\langle M,N\rangle_u^t)_{(t,u)\in\Delta}$ with càdlàg version $\langle M^{s,x}, N^{s,x}\rangle$ under $\mathbb{P}^{s,x}$. Denoting $\langle M\rangle = \langle M,M\rangle$ we also have*

(D.14) $$\langle M,N\rangle = \frac{1}{4}\left(\langle M+N\rangle - \langle M-N\rangle\right).$$

Proof. Let $(s,x) \in [0,T] \times \mathbb{R}^d$. It is clear that $M+N$ and $M-N$ are locally square integrable LMAFs, hence by Proposition D.4 there exist two locally integrable AF $[M+N]$ and $[M-N]$ with respective càdlàg version $[M^{s,x}+N^{s,x}]$ and $[M^{s,x}-N^{s,x}]$ under $\mathbb{P}^{s,x}$.

We recall that, given $\mathcal{N} \in \mathcal{H}^2_{loc}(\mathbb{P}^{s,x})$, the compensator of $[\mathcal{N}]$ is the oblique bracket $\langle \mathcal{N} \rangle$, see e.g. Proposition 4.50, Chapter I in [26]. Consequently, by Proposition D.5 applied with $B = [M+N]$ (resp. $B = [M-N]$), there exists an increasing locally integrable AF $\langle M+N \rangle$ (resp. $\langle M-N \rangle$) with càdlàg version $\langle M^{s,x} + N^{s,x} \rangle$ (resp. $\langle M^{s,x} - N^{s,x} \rangle$) under $\mathbb{P}^{s,x}$. We then define the AF $\langle M, N \rangle$ by (D.14), which by polarization has càdlàg version $\langle M^{s,x}, N^{s,x} \rangle$ under $\mathbb{P}^{s,x}$. □

Remark D.7. *We denote $Pos(\langle M,N \rangle) := \frac{1}{4}\langle M+N \rangle$ and $Neg(\langle M,N \rangle) := \frac{1}{4}\langle M-N \rangle$ the increasing AF with respective càdlàg version $\frac{1}{4}\langle M^{s,x} + N^{s,x} \rangle$ and $\frac{1}{4}\langle M^{s,x} - N^{s,x} \rangle$ appearing in decomposition* (D.14).

The following result can be seen as an extension of Theorem 5 in [33] to time-dependent LMAF. The proof is inspired by the proof of Lemma 4.7 in [4]. First of all we recall that given a local martingale, there exist a continuous local martingale M^c and a purely discontinuous local martingale M^d (vanishing at zero) such that $M = M^c + M^d$, see Theorem 4.18, Chapter I in [26].

Proposition D.8. *Let M be an LMAF with càdlàg version $M^{s,x}$ under $\mathbb{P}^{s,x}$ for $(s,x) \in [0,T] \times \mathbb{R}^d$, and let $M^{s,x,c}$ (resp. $M^{s,x,d}$) be the continuous (resp. purely discontinuous) component of $M^{s,x}$. There exists an LMAF M^d (resp. a locally square integrable LMAF M^c) with càdlàg version $M^{s,x,d}$ (resp. continuous $M^{s,x,c}$) under $\mathbb{P}^{s,x}$.*

Proof. In this proof, $\mu^{s,x}$ (resp. $\nu^{s,x}$)) will denote the jump measure (resp. its compensator), of $M^{s,x}$. Let $0 \le t \le u \le T$ be fixed. We split the proof in two steps.

1. The first step of the proof consists in building a càdlàg version $r \in [t,u] \mapsto \tilde{M}^t_r$ of the random function $r \in [t,u] \mapsto M^t_r$ verifying the following:

   i) $\tilde{M}^t_r$ is $\mathcal{F}_{t,u}$-measurable for all $r \in [t,u]$;

   ii) $\tilde{M}^t_\cdot = M^{s,x}_\cdot - M^{s,x}_t$ $\mathbb{P}^{s,x}$-a.s. for all $(s,x) \in [0,t] \times \mathbb{R}^d$.

   For all $r \in [t,u[$, since $(M^t_r)_+$ is non-negative, we can define $\bar{M}^{t,+}_r := \liminf_{\substack{v \downarrow r \\ v \in [t,u] \cap \mathbb{Q}}} (M^t_v)_+$ and we set $\bar{M}^{t,+}_u := M^t_u$, the final time. Notice that $\bar{M}^{t,+}$ has right-continuous paths by construction and it is $(\mathcal{F}_{t,v})$-progressively measurable by Theorem 17, Chapter IV of [16], being the corresponding filtration right-continuous. We prove just below that it is $\mathbb{P}^{s,x}$-indistinguishable from $(M^{s,x}_\cdot - M^{s,x}_t)_+$ for all $(s,x) \in [0,t] \times \mathbb{R}^d$. Indeed let (s,x) be fixed. By definition of M, there exists a $\mathbb{P}^{s,x}$-null set $\mathcal{N}$ such that for all $v \in [t,u] \in \mathbb{Q}$ and $\omega \in \mathcal{N}^c$, $M^t_v(\omega) = M^{s,x}_v(\omega) - M^{s,x}_t(\omega)$. Then for $\omega \in \mathcal{N}^c$ it holds

$$
\begin{aligned}
\bar{M}^{t,+}_r(\omega) &= \liminf_{\substack{v \downarrow r \\ v \in [t,u] \cap \mathbb{Q}}} \left(M^t_v\right)_+ \\
&= \liminf_{\substack{v \downarrow r \\ v \in [t,u] \cap \mathbb{Q}}} \left(M^{s,x}_v(\omega) - M^{s,x}_t(\omega)\right)_+ \\
&= \left(M^{s,x}_r(\omega) - M^{s,x}_t(\omega)\right)_+ ,
\end{aligned}
$$

   where we used the right-continuity of $M^{s,x}(\omega)$ for the last equality. Now let again $r \in [t,u]$. Then M^t_v is $\mathcal{F}_{t,v}$-measurable for all $v \in [r,u]$ and since the filtration $(\mathcal{F}_t)$ is right-continuous, $\bar{M}^{t,+}_r$ is $\mathcal{F}_{t,r}$-measurable. The exact same construction provides a càdlàg process $\bar{M}^{t,-}$ such that for all $r \in [t,u]$, $\bar{M}^{t,-}_r$ is $\mathcal{F}_{t,r}$-measurable and $\bar{M}^{t,-}$ is $\mathbb{P}^{s,x}$-indistinguishable from $(M^{s,x}_\cdot - M^{s,x}_t)_-$ for all $(s,x) \in [0,t] \times \mathbb{R}^d$. We now set $\bar{M}^t := \bar{M}^{t,+} - \bar{M}^{t,-}$. From what precedes, $\bar{M}^t_r$ is $\mathcal{F}_{t,r}$-measurable for all $r \in [t,u]$ and the process $\bar{M}^t$ is $\mathbb{P}^{s,x}$-indistinguishable from $M^{s,x}_\cdot - M^{s,x}_t$ for all $(s,x) \in [0,t] \times \mathbb{R}^d$. By Theorem 18 b), Chapter IV in [16], it follows that the set

$$
W'_{t,u} := \{\omega \in \Omega \ : \ \text{there is a càdlàg function } h \text{ such that } \bar{M}^t = h \text{ on } [t,u] \cap \mathbb{Q}\}
$$

belongs to $\mathcal{F}_{t,u}$. We now set $\tilde{M}^t := \bar{M}^t 1_{W'_{t,u}}$. The process $\tilde{M}^t$ is càdlàg, $\tilde{M}^t_r$ is $\mathcal{F}_{t,u}$-measurable for all $r \in [t,u]$, even though not necessarily progressively measurable. Yet $\tilde{M}^t$ is still $\mathbb{P}^{s,x}$-indistinguishable from $M^{s,x}_{\cdot} - M^{s,x}_t$ for all $(s,x) \in [0,t] \times \mathbb{R}^d$. Indeed $\bar{M}^t$ is $\mathbb{P}^{s,x}$-indistinguishable from the càdlàg process $M^{s,x}_{\cdot} - M^{s,x}_t$, yielding $1_{W'_{t,u}} = 1$ $\mathbb{P}^{s,x}$-a.s. This concludes the first step.

2. We are now ready to prove the existence of the AF M^c and M^d. For $n \geq 1$, let

$$(N^+_n)^t_u := \sum_{t<r\leq u} \Delta\tilde{M}^t_r 1_{\{\Delta\tilde{M}^t_r \geq 1/n\}}. \tag{D.15}$$

As for all $(s,x) \in [0,t] \times \mathbb{R}^d$, $\tilde{M}^t$ is $\mathbb{P}^{s,x}$-indistinguishable from $M^{s,x}_{\cdot} - M^{s,x}_t$, it holds that

$$(N^+_n)^t_u = \sum_{t<r\leq u} \Delta M^{s,x}_r 1_{\{\Delta M^{s,x}_r \geq 1/n\}} \quad \mathbb{P}^{s,x}\text{-a.s.}$$

Moreover, since $\tilde{M}^t_r$ is $\mathcal{F}_{t,u}$-measurable for all $r \in [t,u]$, $(N^+_n)^t_u$ is also $\mathcal{F}_{t,u}$-measurable. Hence $N^+_n = ((N^+_n)^t_u)_{(t,u)\in\Delta}$ is an increasing AF. Notice that as $M^{s,x}$ is a local martingale, it is a special semi-martingale and by Proposition 2.29, Chapter II in [26], it holds that the process $(|q| 1_{\{|q|\geq 1/n\}}) * \nu^{s,x}$ is locally integrable, which in turn implies that $\sum_{s\leq \cdot} \Delta M^{s,x}_r 1_{\{\Delta M^{s,x}_r \geq 1/n\}} = (q 1_{\{q\geq 1/n\}}) * \mu^{s,x}$ is locally integrable and N^+_n is an increasing locally integrable AF. We apply Proposition D.5 to $B = N^+_n$, which provides the locally integrable AF A, that we will denote by $N^{+,p}_n$. Replacing $\tilde{M}^t$ by $-\tilde{M}^t$ in (D.15), we similarly build two decreasing locally integrable AF N^-_n and $N^{-,p}_n$. We now set $M^d_n := (N^+_n - N^{+,p}_n) - (N^-_n - N^{-,p}_n)$. Under $\mathbb{P}^{s,x}$,

$$N^+_n - N^{+,p}_n = \int_{]s,\cdot]\times\mathbb{R}^d} q 1_{\{q\geq 1/n\}} \mu^{s,x} - \int_{]s,\cdot]\times\mathbb{R}^d} q 1_{\{q\geq 1/n\}} \nu^{s,x} = \int_{]s,\cdot]\times\mathbb{R}^d} q 1_{\{q\geq 1/n\}} (\mu^{s,x} - \nu^{s,x})(dr,dq)$$

and

$$N^-_n - N^{-,p}_n = \int_{]s,\cdot]\times\mathbb{R}^d} q 1_{\{q\leq -1/n\}} \mu^{s,x} - \int_{]s,\cdot]\times\mathbb{R}^d} q 1_{\{q\geq 1/n\}} \nu^{s,x} = \int_{]s,\cdot]\times\mathbb{R}^d} q 1_{\{q\leq -1/n\}} (\mu^{s,x} - \nu^{s,x})(dr,dq).$$

Hence M^d_n is a LMAF whose càdlàg version is

$$(M^d_n)^{s,x} := \int_{]s,\cdot]\times\mathbb{R}^d} q 1_{\{|q|\geq 1/n\}} (\mu^{s,x} - \nu^{s,x})(dr,dq)$$

under $\mathbb{P}^{s,x}$. For all $0 \leq s \leq t \leq u \leq T$, $\left(M^d_n\right)^t_u$ converges towards

$$(M^d_u)^{s,x} - (M^d_t)^{s,x} := \int_{]s,\cdot]\times\mathbb{R}^d} q (\mu^{s,x} - \nu^{s,x})(dr,dq),$$

in probability under $\mathbb{P}^{s,x}$. Indeed previous limit stochastic integral is well-defined since $q 1_{\{|q|\geq 1/n\}}$ converges in $\mathcal{L}^2(\nu^{s,x})$ to q.

By Lemma D.3 there exists an $\mathcal{F}_{t,u}$-measurable random variable $\left(M^d\right)^t_u$ such that for all $(s,x) \in [0,t] \times \mathbb{R}^d$, $\left(M^d\right)^t_u = M^{s,x,d}_u - M^{s,x,d}_t$ $\mathbb{P}^{s,x}$-a.s. Hence M^d is a LMAF with càdlàg version $M^{s,x,d}$ under $\mathbb{P}^{s,x}$. We conclude the proof by setting $M^c := M - M^d$.

□

This is Proposition 4.16 in [4].

Proposition D.9. *Let $\mathcal{H}^{2,dt} := \{M \in \mathcal{H}^2_0 \ : \ d\langle M\rangle \ll dt\}$ and $\mathcal{H}^{2,\perp dt} := \{M \in \mathcal{H}^2_0 \ : \ d\langle M\rangle \perp dt\}$. $\mathcal{H}^{2,dt}$ and $\mathcal{H}^{2,\perp dt}$ are orthogonal sub-Hilbert spaces of $\mathcal{H}^2_0$, $\mathcal{H}^2_0 = \mathcal{H}^{2,dt} \oplus \mathcal{H}^{2,\perp dt}$, and any element M of $\mathcal{H}^{2,dt}_{loc}$ is strongly orthogonal to any element N of $\mathcal{H}^{2,\perp dt}_{loc}$, i.e. $\langle M, N\rangle = 0$.*

Finally we can generalize the main result of [4], i.e. Proposition 4.17.

Proposition D.10. *Let M, N be two locally square integrable LMAFs. Assume that the AF $\langle N \rangle$ is absolutely continuous w.r.t. the Lebesgue measure dt. There exists a unique (up to a zero-potential set) function $w \in \mathcal{B}([0,T] \times \mathbb{R}^d)$ such that for any $(s,x) \in [0,T] \times \mathbb{R}^d$, $\langle M^{s,x}, N^{s,x} \rangle = \int_s^{\cdot \vee s} w(r, X_r)dr$.*

Proof. Given a square integrable martingale $\mathcal{N}$ we denote by $\mathcal{N}^{dt}$ its orthogonal projection on $\mathcal{H}^{2,dt}$.

Let $\langle M, N \rangle$ be the bounded variation AF given by Corollary D.6 and $Pos(\langle M, N \rangle)$ and $Neg(\langle M, N \rangle)$ be the corresponding increasing AF given by Remark D.7. Let $(s,x) \in [0,T] \times \mathbb{R}^d$. Let $(\tau_n)_{n \geq 1}$ be a sequence of stopping times such that $M^{s,x,\tau_n}, N^{s,x,\tau_n} \in \mathcal{H}_0^2(\mathbb{P}^{s,x})$. As $\langle N^{s,x,\tau_n} \rangle = \langle N^{s,x} \rangle^{\tau_n}$, it is immediate that $d\langle N^{s,x,\tau_n} \rangle \ll dt$. Hence $N^{s,x,\tau_n} \in \mathcal{H}_0^{2,dt}$ and by Proposition D.9,

$$\begin{aligned}\langle M^{s,x,\tau_n}, N^{s,x,\tau_n} \rangle &= \left\langle (M^{s,x,\tau_n})^{dt}, N^{s,x,\tau_n} \right\rangle \\ &= \frac{1}{4} \left\langle (M^{s,x,\tau_n})^{dt} + N^{s,x,\tau_n} \right\rangle - \frac{1}{4} \left\langle (M^{s,x,\tau_n})^{dt} - N^{s,x,\tau_n} \right\rangle.\end{aligned}$$

Consequently

$$Pos\left(\langle M^{s,x,\tau_n}, N^{s,x,\tau_n} \rangle\right) = \frac{1}{4} \left\langle (M^{s,x,\tau_n})^{dt} + N^{s,x,\tau_n} \right\rangle$$

and

$$Neg\left(\langle M^{s,x,\tau_n}, N^{s,x,\tau_n} \rangle\right) = \frac{1}{4} \left\langle (M^{s,x,\tau_n})^{dt} - N^{s,x,\tau_n} \right\rangle.$$

Recall that $N^{s,x,\tau_n} \in \mathcal{H}_0^{2,dt}$ and that $\mathcal{H}_0^{2,dt}$ is a vector space. Consequently, by polarization we have $dPos\left(\langle M^{s,x,\tau_n}, N^{s,x,\tau_n} \rangle\right) \ll dt$ and $dNeg\left(\langle M^{s,x,\tau_n}, N^{s,x,\tau_n} \rangle\right) \ll dt$. Moreover, it holds

$$Pos\left(\langle M^{s,x}, N^{s,x} \rangle\right) = \sum_{n \geq 1} Pos\left(\langle M^{s,x,\tau_n}, N^{s,x,\tau_n} \rangle\right) 1_{]\tau_n, \tau_{n+1}]} \tag{D.16}$$

and

$$Neg\left(\langle M^{s,x}, N^{s,x} \rangle\right) = \sum_{n \geq 1} Neg\left(\langle M^{s,x,\tau_n}, N^{s,x,\tau_n} \rangle\right) 1_{]\tau_n, \tau_{n+1}]}. \tag{D.17}$$

It follows from (D.16) and (D.17) and what preceeds that $dPos\left(\langle M^{s,x}, N^{s,x} \rangle\right) \ll dt$ as well as $dNeg\left(\langle M^{s,x}, N^{s,x} \rangle\right) \ll dt$ for all $(s,x) \in [0,T] \times \mathbb{R}^d$. We recall that $Pos(\langle M, N \rangle)$ (resp. $Neg(\langle M, N \rangle)$) is an increasing AF with càdlàg version $Pos\left(\langle M^{s,x}, N^{s,x} \rangle\right)$ (resp. $Neg\left(\langle M^{s,x}, N^{s,x} \rangle\right)$). Then Proposition 4.13 in [4] ensures the existence of two functions w_+ and w_- in $\mathcal{B}([0,T] \times \mathbb{R}^d)$ such that for any $(s,x) \in [0,T] \times \mathbb{R}^d$, $Pos\left(\langle M^{s,x}, N^{s,x} \rangle\right) = \int_s^{\cdot \vee s} w_+(r, X_r)dr$ and $Neg\left(\langle M^{s,x}, N^{s,x} \rangle\right) = \int_s^{\cdot \vee s} w_-(r, X_r)dr$. Setting $w = w_+ - w_-$, by additivity, we conclude the proof. □

We come back to the notion of Markov martingale domain $\mathcal{D}(\mathfrak{a})$ introduced in Definition 5.5.

Remark D.11. *Let $\phi \in \mathcal{D}(\mathfrak{a})$. We set*

$$M[\phi]_u^t := \phi(u, X_u) - \phi(t, X_t) - \int_u^t \mathfrak{a}(\phi)(r, X_r)dr, \ (t,u) \in \Delta. \tag{D.18}$$

Obviously $(M[\phi]_u^t)_{(t,u) \in \Delta}$ is a LMAF with càdlàg version

$$M[\phi]^{s,x} = 1_{[s,T]}(\cdot) \left(\phi(\cdot, X_\cdot) - \phi(s,x) - \int_s^\cdot \mathfrak{a}(\phi)(r, X_r)dr \right)$$

under $\mathbb{P}^{s,x}$ for all $(s,x) \in [0,T] \times \mathbb{R}^d$.

The following corollary is the main result of this section.

Corollary D.12. *Let ϕ and ψ be two elements of the Markov martingale domain $\mathcal{D}(\mathfrak{a})$ such that $M[\phi]^{s,x} \in \mathcal{H}^2_{loc}(\mathbb{P}^{s,x})$ and $d\langle M[\phi]^{s,x}\rangle \ll dt$ for all $(s,x) \in [0,T] \times \mathbb{R}^d$.*

1. *There exists a unique (up to a zero-potential set) measurable function $\Gamma_c(\phi,\psi) \in \mathcal{B}([0,T]\times\mathbb{R}^d)$ such that for all $(s,x) \in [0,T]\times\mathbb{R}^d$,*

$$\langle M[\phi]^{s,x,c}, M[\psi]^{s,x,c}\rangle_\cdot = \int_s^\cdot \Gamma_c(\phi,\psi)(r,X_r)dr. \tag{D.19}$$

2. *Assume moreover that $M[\psi]^{s,x} \in \mathcal{H}^2_{loc}(\mathbb{P}^{s,x})$ for all $(s,x) \in [0,T]\times\mathbb{R}^d$. There exists a unique (up to a zero-potential set) measurable function $\Gamma(\phi,\psi) \in \mathcal{B}([0,T]\times\mathbb{R}^d)$ such that for all $(s,x) \in [0,T]\times\mathbb{R}^d$,*

$$\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle_\cdot = \int_s^\cdot \Gamma(\phi,\psi)(r,X_r)dr. \tag{D.20}$$

Proof. Let $(M[\phi]^t_u)_{(t,u)\in\Delta}$ and $(M[\psi]^t_u)_{(t,u)\in\Delta}$ be the LMAFs with respective càdlàg version $M[\phi]^{s,x}$ and $M[\psi]^{s,x}$ under $\mathbb{P}^{s,x}$ defined in (D.18). By assumption $M[\phi]^{s,x} \in \mathcal{H}^2_{loc}(\mathbb{P}^{s,x})$ and $d\langle M[\phi]^{s,x}\rangle \ll dt$. Hence the random field $(M[\phi]^t_u)_{(t,u)\in\Delta}$ is a locally square integrable LMAF such that $d\langle M[\phi]\rangle \ll dt$, recalling that $\langle M[\phi]\rangle$ was provided by Lemma D.6.

By Proposition D.8 there exist two locally square integrable LMAF $M[\phi]^c$ and $M[\psi]^c$ with respective càdlàg version $M[\phi]^{s,x,c}$ and $M[\psi]^{s,x,c}$ under $\mathbb{P}^{s,x}$. Since for all $(s,x)\in[0,T]\times\mathbb{R}^d$,

$$\langle M[\phi]^{s,x}\rangle = \langle M[\phi]^{s,x,c} + M[\phi]^{s,x,d}\rangle = \langle M[\phi]^{s,x,c}\rangle + \langle M[\phi]^{s,x,d}\rangle,$$

$d\langle M[\phi]\rangle \ll dt$ implies that $d\langle M[\phi]^c\rangle \ll dt$ (and $d\langle M[\phi]^d\rangle \ll dt$). Now item 1. follows from Proposition D.10.

As far as item 2. is concerned, assume now that $M[\psi]^{s,x} \in \mathcal{H}^2_{loc}(\mathbb{P}^{s,x})$ for all $(s,x)\in[0,T]\times\mathbb{R}^d$. Then, by definition, $(M[\psi]^t_u)_{(t,u)\in\Delta}$ is a locally square integrable LMAF and existence of $\Gamma(\phi,\psi)$ is given again by Proposition D.10.

□

Appendix E: Extension to mean-field optimization

This short section is devoted to the proof of the equivalence between the optimization problems (1.5) and (1.6).

Lemma E.1. *Let $F:\mathbb{R}\to\mathbb{R}$ be a convex differentiable function. Then $\mathbb{Q}^*\in\mathcal{P}(\Omega)$ is solution to Problem* (1.5) *if and only if $\mathbb{Q}^*$ is solution to the linearized Problem* (1.6)*.*

Proof. Assume first that $\mathbb{Q}^*$ is solution to Problem (1.6). Since F is convex differentiable, for all $x,y\in\mathbb{R}^d$, it holds that $F(x)-F(y)\geq F'(y)(x-y)$. Applying this inequality to $x=\mathbb{E}^{\mathbb{Q}}[\varphi(X)]$ and $y=\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]$ for some $\mathbb{Q}\in\mathcal{P}(\Omega)$ yields

$$F(\mathbb{E}^{\mathbb{Q}}[\varphi(X)]) - F(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]) \geq F'(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)])(\mathbb{E}^{\mathbb{Q}}[\varphi(X)] - \mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]).$$

Adding $H(\mathbb{Q}|\mathbb{P}) - H(\mathbb{Q}^*|\mathbb{P})$ to each sides of the previous inequality, we get

$$F(\mathbb{E}^{\mathbb{Q}}[\varphi(X)]) + H(\mathbb{Q}|\mathbb{P}) - F(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]) - H(\mathbb{Q}^*|\mathbb{P}) \geq \mathbb{E}^{\mathbb{Q}}[\tilde\varphi(X)] + H(\mathbb{Q}|\mathbb{P}) - \mathbb{E}^{\mathbb{Q}^*}[\tilde\varphi(X)] - H(\mathbb{Q}^*|\mathbb{P}) \geq 0,$$

where we used the fact that $\mathbb{Q}^*$ is solution of Problem (1.6) for the right-most inequality. Hence for all $\mathbb{Q}\in\mathcal{P}(\Omega)$,

$$F(\mathbb{E}^{\mathbb{Q}}[\varphi(X)]) + H(\mathbb{Q}|\mathbb{P}) \geq F(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]) + H(\mathbb{Q}^*|\mathbb{P}),$$

and $\mathbb{Q}^*$ is solution to Problem (1.5).

Conversely, assume that $\mathbb{Q}^*$ is solution to Problem (1.5). Let $\lambda \in [0,1]$. By definition, for all $\mathbb{Q} \in \mathcal{P}(\Omega)$,

$$F\left(\mathbb{E}^{\lambda\mathbb{Q}+(1-\lambda)\mathbb{Q}^*}[\varphi(X)]\right) + H(\lambda\mathbb{Q} + (1-\lambda)\mathbb{Q}^*|\mathbb{P}) - F\left(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right) - H(\mathbb{Q}^*|\mathbb{P}) \geq 0,$$

that is

$$F\left(\lambda\mathbb{E}^{\mathbb{Q}}[\varphi(X)] + (1-\lambda)\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right) - F\left(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right) + H(\lambda\mathbb{Q} + (1-\lambda)\mathbb{Q}^*|\mathbb{P}) - H(\mathbb{Q}^*|\mathbb{P}) \geq 0. \tag{E.1}$$

By the convexity of the relative entropy, see Remark 2.5 item 1., we have

$$H(\lambda\mathbb{Q} + (1-\lambda)\mathbb{Q}^*|\mathbb{P}) - H(\mathbb{Q}^*|\mathbb{P}) \leq \lambda(H(\mathbb{Q}|\mathbb{P}) - H(\mathbb{Q}^*|\mathbb{P})). \tag{E.2}$$

Combining (E.1) and (E.2) and dividing by λ we get

$$\frac{1}{\lambda}\left(F\left(\lambda\mathbb{E}^{\mathbb{Q}}[\varphi(X)] + (1-\lambda)\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right) - F\left(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right)\right) + H(\mathbb{Q}|\mathbb{P}) - H(\mathbb{Q}^*|\mathbb{P}) \geq 0,$$

and letting $\lambda \to 0$ yields

$$F'\left(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right)(\mathbb{E}^{\mathbb{Q}}[\varphi(X)] - \mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]) + H(\mathbb{Q}|\mathbb{P}) - H(\mathbb{Q}^*|\mathbb{P}) \geq 0,$$

which rewrites

$$F'\left(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right)\mathbb{E}^{\mathbb{Q}}[\varphi(X)] + H(\mathbb{Q}|\mathbb{P}) \geq F'\left(\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)]\right)\mathbb{E}^{\mathbb{Q}^*}[\varphi(X)] + H(\mathbb{Q}^*|\mathbb{P}).$$

We conclude from the previous inequality that $\mathbb{Q}^*$ is a solution of Problem (1.6). □

Funding

The research of the first named author is supported by a doctoral fellowship PRPhD 2021 of the Région Île-de-France. The research of the second and third named authors was partially supported by the ANR-22-CE40-0015-01 project (SDAIM).

References

[1] Bahouri, H., Chemin, J.-Y., and Danchin, R. *Fourier analysis and nonlinear partial differential equations*, vol. 343 of *Grundlehren Math. Wiss.* Berlin: Heidelberg, 2011.

[2] Bandini, E., and Russo, F. Weak Dirichlet processes with jumps. *Stochastic Processes Appl. 127*, 12 (2017), 4139–4189.

[3] Bandini, E., and Russo, F. Weak Dirichlet processes and generalized martingale problems. *Stochastic Processes Appl. 170* (2024), 37. Id/No 104261.

[4] Barrasso, A., and Russo, F. A Note on Time-Dependent Additive Functionals. *Communications on Stochastic Analysis 11*, 3 (2017), 313–334.

[5] Barrasso, A., and Russo, F. Backward Stochastic Differential Equations with no driving martingale, Markov Processes and Associated Pseudo-Partial Differential Equations: Part II: Decoupled Mild Solutions and Examples. *Journal of Theoretical Probability 34*, 3 (2021), 1110–1148.

[6] Barrasso, A., and Russo, F. Martingale driven BSDEs, PDEs and other related deterministic problems. *Stochastic Processes and their Applications 133* (2021), 193–228.

[7] Barrasso, A., and Russo, F. Backward Stochastic Differential Equations with No Driving Martingale and Pseudo-PDEs. *Journal of Stochastic Analysis 3*, 1 (2022), 3.

[8] Bierkens, J., and Kappen, H. J. Explicit solution of relative entropy weighted control. *Syst. Control Lett. 72* (2014), 36–43.

[9] Boué, M., and Dupuis, P. A variational representation for certain functionals of Brownian motion. *Ann. Probab. 26*, 4 (1998), 1641–1659.

[10] Bourdais, T., Oudjane, N., and Russo, F. An entropy penalized approach for stochastic control problem (Complete Version). *Preprint hal-04193113* (2023).

[11] BRUNICK, G., AND SHREVE, S. Mimicking an Itô process by a solution of a stochastic differential equation. *The Annals of Applied Probability 23*, 4 (2013), 1584–1628.

[12] CAMMARDELLA, N., BUŠIĆ, A., AND MEYN, S. P. Kullback-Leibler-quadratic optimal control. *SIAM J. Control Optim. 61*, 5 (2023), 3234–3258.

[13] CAMMARDELLA, N., BUŠIĆ, A., JI, Y., AND MEYN, S. Kullback-Leibler-Quadratic Optimal Control of Flexible Power Demand. In *2019 IEEE 58th Conference on Decision and Control (CDC)* (2019), pp. 4195–4201.

[14] CLAISSE, J., CONFORTI, G., REN, Z., AND WANG, S. Mean Field Optimization Problem Regularized by Fisher Information. *Arxiv:2302.05938* (2023).

[15] DELLACHERIE, C. Sur les noyaux sigma-finis. Seminaire de probabilites XV, Univ. Strasbourg 1979/80, Lect. Notes Math. 850, 371-387 (1981)., 1981.

[16] DELLACHERIE, C., AND MEYER, P.-A. *Probabilités et potentiel. Chap. I à IV. Ed. entièrement refondue*, vol. 15 of *Publ. Inst. Math. Univ. Strasbourg*. Hermann, Paris, 1975.

[17] DELLACHERIE, C., AND MEYER, P.-A. *Probabilités et potentiel. Chapitres V à VIII: Théorie des martingales. Ed. ent. ref.*, vol. 17 of *Publ. Inst. Math. Univ. Strasbourg*. Hermann, Paris, 1980.

[18] DUNFORD, N., AND SCHWARTZ, J. T. *Linear Operators. I. General Theory*. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.

[19] DUPUIS, P., AND ELLIS, R. S. *A weak convergence approach to the theory of large deviations*. Wiley Ser. Probab. Stat. Chichester: John Wiley & Sons, 1997.

[20] EL KAROUI, N., NGUYEN, D., AND JEANBLANC-PICQUÉ, M. Compactification methods in the control of degenerate diffusions: Existence of an optimal control. *Stochastics 20* (1987), 169–219.

[21] FLEMING, W. H. Logarithmic transformations and stochastic control. Advances in filtering and optimal stochastic control, Proc. IFIP-WG 7/1 Work. Conf., Cocoyoc/Mex. 1982, Lect. Notes Contr. Inf. Sci. 42, 131-141 (1982)., 1982.

[22] FLEMING, W. H., AND MITTER, S. K. Optimal control and nonlinear filtering for nondegenerate diffusion processes. *Stochastics 8* (1982), 63–77.

[23] HAUSSMANN, U. G. Existence of optimal Markovian controls for degenerate diffusions. Stochastic differential systems, Proc. 3rd Bad Honnef Conf. 1985, Lect. Notes Control Inf. Sci. 78, 171-186, 1986.

[24] HAUSSMANN, U. G., AND LEPELTIER, J. P. On the existence of optimal controls. *SIAM J. Control Optim. 28*, 4 (1990), 851–902.

[25] ISSOGLIO, E., AND RUSSO, F. Stochastic differential equations with singular coefficients: The martingale problem view and the stochastic dynamics view. *Journal of Theoretical Probability* (2024), 1–42.

[26] JACOD, J., AND SHIRYAEV, A. N. *Limit theorems for stochastic processes*, second ed., vol. 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.

[27] KARATZAS, I., AND SHREVE, S. E. *Brownian motion and stochastic calculus*, second ed., vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[28] KOMATSU, T. Markov processes associated with certain integro-differential operators. *Osaka J. Math. 10* (1973), 271–303.

[29] KOMATSU, T. On the martingale problem for generators of stable processes with perturbations. *Osaka J. Math. 21* (1984), 113–132.

[30] LÉONARD, C. Girsanov theory under a finite entropy condition. In *Séminaire de probabilités XLIV*. Berlin: Springer, 2012, pp. 429–465.

[31] LIU, L., MAJKA, M. B., AND SZPRUCH, Ł. Polyak-Łojasiewicz inequality on the space of measures and convergence of mean-field birth-death processes. *Appl. Math. Optim. 87*, 3 (2023), 27. Id/No 48.

[32] MARINELLI, C., AND RÖCKNER, M. On the maximal inequalities of Burkholder, Davis and Gundy. *Expo. Math. 34* (2016), 1–36.

[33] MEYER, P. A. Martingales locales fonctionnelles additives. I. Semin. Probab. XII, Univ. Strasbourg 1976/77, Lect. Notes Math. 649, 775-785 (1978)., 1978.

[34] PALMOWSKI, Z., AND ROLSKI, T. A technique for exponential change of measure for Markov processes. *Bernoulli 8*, 6 (2002), 767–785.

[35] RUSSO, F., AND VALLOIS, P. The generalized covariation process and Itô formula. *Stochastic Process. Appl. 59*, 1 (1995), 81–104.

[36] RUSSO, F., AND VALLOIS, P. *Stochastic Calculus via Regularizations*, vol. 11. Springer International Publishing, 2022.

[37] SANOV, I. N. *On the probability of large deviations of random variables*, vol. 42. Matematicheskii Sbornik. Novaya Seriya, 1957.

[38] SEGURET, A., ALASSEUR, C., BONNANS, J. F., DE PAOLA, A., OUDJANE, N., AND TROVATO, V. Decomposition of convex high dimensional aggregative stochastic control problems. *Appl. Math. Optim. 88*, 1 (2023), 35. Id/No 8.

[39] SÉGURET, A., LE CORRE, T., AND OUDJANE, N. A decentralized algorithm for a mean field control problem of piecewise

deterministic Markov processes. *ESAIM, Probab. Stat. 28* (2024), 22–45.

[40] STROOCK, D. W. Diffusion processes associated with Lévy generators. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 32*, 3 (1975), 209–244.

[41] STROOCK, D. W., AND VARADHAN, S. R. S. Diffusion processes with continuous coefficients. I. *Comm. Pure Appl. Math. 22* (1969), 345–400.

[42] STROOCK, D. W., AND VARADHAN, S. R. S. Diffusion processes with continuous coefficients. II. *Comm. Pure Appl. Math. 22* (1969), 479–530.

[43] STROOCK, D. W., AND VARADHAN, S. R. S. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[44] THEODOROU, E., BUCHLI, J., AND SCHAAL, S. Reinforcement learning of motor skills in high dimensions: A path integral approach. In *2010 IEEE International Conference on Robotics and Automation* (2010), pp. 2397–2403.

[45] THIJSSEN, S., AND KAPPEN, H. Path integral control and state-dependent feedback. *Physical Review E 91*, 3 (2015), 032104.