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Stochastic Calculus via Regularizations

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To Cristina and to the Dark Ladies Mara and Michela. To the memory of my parents (FR)

To Marianne, Laure, Isabelle, and Nicolas and to the memory of my parents (PV)

Preface

In physics, classical analysis plays a central role. For instance in Newtonian mechanics, thermodynamics, and electricity, many phenomena are well explained by deterministic models involving either ordinary differential equations or partial differential equations.

In the deterministic world, theoretically, the future evolution of a system depends on initial conditions and on some parameters driving its own dynamics; nevertheless, in practice, their values are often measured with uncertainty. Moreover, the theory of chaos explains that small variations in the initial data can generate big fluctuations later on. In addition, deterministic models used for representing complex phenomena currently require a huge number of equations which are either difficult or impossible to solve within a reasonable time. In fact complexity and uncertainty can be integrated directly via a probabilistic model. Mention can be made of statistical mechanics, networks (Internet, gene interaction), financial markets, and percolation. In probability theory, a random quantity which fluctuates as a function depending on time is modeled by a *stochastic process*. On a given probability space $(\Omega, \mathcal{F}, P)$, a process $X = (X_t(\omega), t \in [0, T], \omega \in \Omega)$ is a "measurable" collection of random variables which are defined on the same underlying space and take their values in some space E. A process depends on time t and on the random realization ω. Fixing ω, $t \mapsto X_t(\omega)$ is a real function, often called *path* or *trajectory* of the process. In the sequel, the process X will be denoted either (X_t) or $(X(t))$, omitting the variable ω. For the sake of simplicity, in this introduction, we essentially only deal with $E = \mathbb{R}$.

In deterministic analysis, differentiation and integration play an important role. Consider two functions $f, x : \mathbb{R} \to \mathbb{R}$ which are assumed to be differentiable. Thus, the derivative of $t \mapsto f\big(x(t)\big)$ at time t is $f'\big(x(t)\big)x'(t)$. This property is equivalent to

$$f\big(x(t)\big) = f\big(x(0)\big) + \int_0^t f'\big(x(s)\big)x'(s)ds = \int_0^t f'\big(x(s)\big)dx(s), t \in [0, T], \quad (1)$$

where the latter integral is intended in the sense of Lebesgue-Stieltjes.

Now, let $\big((X(t), t \in [0, T]\big)$ be a real valued stochastic process. Fixing ω, the path $t \mapsto X(t, \omega)$ is generally continuous but is neither differentiable nor with bounded variation as it happens in particular when X is a Brownian motion. Therefore, even if f is C^1, (1) cannot be applied since $t \mapsto f\big(X(t)\big)$ is not differentiable. An important field in probability theory is the so called *stochastic calculus* which combines probability and calculus, and allows in particular

1. giving a sense to integrals (in time) of the type $\int_0^t f'\big(X(s)\big)dX(s)$,
2. developing a useful and efficient *(stochastic) calculus*, i.e., a formula for *differentiating* $f\big(X(t)\big)$ as (1),

for a large class of non-differentiable processes X and functions f.

In fact, for a given *integrator process* $\big(X_t, t \in [0, T]\big)$, more generally, one goal of *stochastic integration* is to define the *stochastic integral* $\int_0^T Y(s)dX(s)$ for a large class of *integrands* $\big(Y_t, t \in [0, T]\big)$ also defined on the same probability space. Obviously, if for almost all ω, the path $t \mapsto X_t(\omega)$ were differentiable and $\int_0^T |Y_t(\omega)\dot{X}_t(\omega)|dt < \infty$, the above-mentioned stochastic integral could be defined as the Lebesgue integral in time for almost all ω. As mentioned earlier, most of the significant processes intervening in classical stochastic models are not differentiable, so that the previous scheme does not permit defining $\int_0^T Y(s)dX(s)$.

Classical stochastic integration is a nice combination of Lebesgue integration and martingale theory. The most famous (stochastic) integral is the *Itô* integral which is defined for instance when the integrator is the Brownian motion $X = W$. However, modern (Itô's) stochastic integration can be developed in a more general setting, which refers to the case when X is a *semimartingale* with respect to an underlying filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e., the sum of a (local) $(\mathcal{F}_t)$-martingale M and an $(\mathcal{F}_t)$-"adapted" (i.e., non-anticipative) bounded variation process V. Here, we essentially deal with continuous semimartingales, i.e., when M and V are continuous processes. If Y and X are both semimartingales, another celebrated integral is the so called *Fisk-Stratonovich integral*, denoted by $\int_0^t Y \circ dX$ which coincides with the sum of Itô's integral $\int_0^t YdX$ and half the (oblique) bracket $\langle Y, X\rangle$ between the martingale components of Y and X. When $X = Y$, one also denotes $\langle X\rangle := \langle Y, X\rangle$. Itô's (and Fisk-Stratonovich's) integral only allows non-anticipating integrands Y with respect to the semimartingale integrator X. A process (Y_t) is said to be *adapted* if, for any t, the random variable Y_t is $\mathcal{F}_t$-measurable. In many situations (Y_t) must be progressively measurable, which is a more stringent notion, see Definition 2.9. Time evolution plays a crucial role in dynamical physical systems and most of the interesting (random) quantities Y are adapted to the information carried by X. At time t, this information is concentrated in the σ-field $\mathcal{F}_t$, which is generated by the collection of random variables $(X_u, 0 \leq u \leq t)$. In stochastic models for finance, at each time t, the investor knows the assets (stocks, bonds, interest rates) price (X_t)

in a given market. Thus, the portfolio composition Y of the investor at time t, must be non-anticipating for the increasing family of σ-fields $(\mathcal{F}_t)$. If X is a Brownian motion, $(\mathcal{F}_t)$ is called a *Brownian filtration*.

Given a progressively measurable integrand process (Y_t), and a semimartingale $X = M + V$, the Itô integral $\int_0^t Y_s dX_s$ is defined as the sum of $\int_0^t Y_s dM_s$ and $\int_0^t Y_s dV_s$. The former one makes major use of the martingale property of M and the second integral is defined for any ω as the classical Lebesgue integral under the assumption $\int_0^T |Y_s| d\|V\|_s < \infty$, where $\|V\|$ stands for the total variation process of V. In order to give a sense to the stochastic integral with respect to the martingale M, let us start with a bounded and piecewise constant process of the type $Y_t = Y_{-1} 1_0(t) + \sum_{i=0}^{N-1} Y_i 1_{]t_i, t_{i+1}]}(t), t \in [0, T]$, where Y_i is $\mathcal{F}_{t_i}$-measurable, Y_{-1} is $\mathcal{F}_0$-measurable and $0 = t_0 < \cdots < t_i < \cdots < t_N = T$ is a subdivision of $[0, T]$. It seems reasonable to set

$$\int_0^t Y_s dM_s := Y_{-1} M_0 + Y_0 (M_{t_1} - M_{t_0}) + \cdots + Y_i (M_t - M_{t_i}), \tag{2}$$

for any t in $[t_i, t_{i+1}]$. We denote by $\mathcal{E}$ the class of such elementary processes Y. It can be proved, see Sect. 5.1, that the map I, which to any $Y \in \mathcal{E}$ associates the stochastic integral $\left(\int_0^t Y_s dM_s, t \in [0, T] \right)$ is linear, takes its values in the set of continuous local martingales (see Definition 2.21), where that space of continuous stochastic processes is equipped with the topology of the uniform convergence in probability. Moreover, the map I can be prolonged by continuity to the space of progressively measurable processes Y such that $\int_0^T Y_s^2 d < M >_s < \infty$ almost surely. In the particular case of Brownian motion $M = W$, we have $< W >_t = t$ for any $t \geq 0$.

The class of progressively measurable processes (resp. semimartingales) is the right setting for integrands (resp. integrators). Indeed the map $Y \in \mathcal{E} \mapsto \int_0^{\cdot} Y dX$ is continuous if and only if X is a semimartingale, according to the celebrated Bichteler–Dellacherie theorem, see Section III.7 in [273], or Section IV-2.16 in [276] or Section VIII.4 in [78].

As already pointed out, classical objects of real analysis are ordinary differential equations, which can be expressed in the differential form or equivalently in the integral form. Their stochastic counterpart are the so called *stochastic differential equations* (SDEs). They appear in an integral formulation, since there is no natural path-wise differentiation for stochastic integrators like semimartingales, in particular Brownian motion.

The most celebrated SDEs are those driven by a classical Brownian motion $W = (W_t)$, whose coefficients $a, b : [0, T] \times \mathbb{R} \to \mathbb{R}$ are Borel functions, and an initial value ξ which is a random variable independent of (W_t), i.e.,

$$X_t = \xi + \int_0^t a(s, X_s) dW_s + \int_0^t b(s, X_s) ds, \ t \in [0, T], \tag{3}$$

where $\int_0^t a(s, X_s) dW_s$ is an Itô integral, which is a (local) martingale. In particular a solution is a semimartingale whose value at 0 is ξ, its local martingale part is $\int_0^t a(s, X_s) dW_s$ and its bounded variation component equals $\int_0^t b(s, X_s) ds$. For them, it is often convenient to adopt the improper differential formulation

$$\begin{cases} dX_t = a(t, X_t) dW_t + b(t, X_t) dt \\ X_0 \ = \xi. \end{cases} \tag{4}$$

The solutions of SDEs of the type (4), are called *diffusion processes* or *diffusions*. The family of diffusions processes is an important subclass of the one of semimartingales. In physics, the motion of a microscopic particle in a medium with velocity $b(t, x)$ at time t and position x, subjected to noise perturbation, with intensity a, is often described by a diffusion. In Chaps. 12 and 13 we develop the classical theory of SDEs and mainly study existence and uniqueness. The SDE (4) admits multi-dimensional and infinite dimensional formulations, see [73].

Note that if the *diffusion* coefficient a vanishes then (4) is an ordinary differential equation, for which the differential calculus is the main device. In order to study stochastic differential equations, one needs the aforementioned *Itô's stochastic calculus*, supporting Itô stochastic integration. The central feature of that powerful calculus is the so called *Itô's formula*, which is in fact a *change of variable formula*. That instrument constitutes the stochastic extension of the *fundamental theorem of integral calculus* or chain rule property stated in (1). Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is of class C^2 and X is a semimartingale. Then (1) can be generalized into

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d < X >_s, \quad 0 \le t \le T. \tag{5}$$

If X is a differentiable process, then $< X >= 0$ since its martingale part is constant. In the Stratonovich formulation (5) can be also written as

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \langle f'(X), X \rangle_t, \quad 0 \le t \le T. \tag{6}$$

When $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is of class $C^{1,2}$, then (5) extends to

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_s f(s, X_s)ds + \int_0^t \partial_x f(s, X_s)dX_s + \frac{1}{2}\int_0^t \partial^2_{xx} f(s, X_s)d < X >_s, \quad (7)$$

where $0 \le t \le T$. In Sect. 14.2 we will extend (6) to the case where $f \in C^1$, thus retrieving the fundamental theorem of integral calculus, see also considerations below.

Although Itô's stochastic calculus is a powerful apparatus, it is based on two important prerequisites, mentioned earlier: the fact that the integrand is progressively measurable and the semimartingale property. Those are significant restrictions. We give below several motivations for introducing stochastic integrals in the case when the integrand (Y_t) is either non-adapted (anticipating) or the integrator (X_t) is a not a semimartingale.

1. **The problem of anticipation.** In some circumstances one needs to introduce stochastic integrals, where the integrator S is an $(\mathcal{F}_t)$-semimartingale and the integrand of the type $Y_t := \phi(A, t)$ where $(\phi(x, \cdot))$ is $(\mathcal{F}_t)$-adapted, for any $x \in \mathbb{R}$ and A is a random variable which is not $\mathcal{F}_0$-measurable. In this case (Y_t) is generally non-adapted. This situation arises for instance when $(\phi(x, \cdot))$ is the solution of (3) for $\xi \equiv x$ and A is an initial condition which is not independent of W and $(\mathcal{F}_t)$ is the Brownian filtration generated by W. Let us give another example raising from finance. Let $(S_t)_{0 \le t \le T}$ be an asset price, supposed to be a semimartingale. Let consider an insider who has an information on the price at the maturity T. He will possibly invest on a portfolio whose composition takes into account S_T. Therefore, the cumulated wealth will possibly integrate expressions of the form $\int_0^t \psi(S_T, t)dS_t$, where $\phi(x, t)$ is, for any $x \in \mathbb{R}$, non-anticipating. Indeed many authors have modeled the behavior of an inside trader, starting from the early works of Grorud and Pontier, see [154].

   The question of defining integrals with integrands of the type $Y_t := \phi(A, t)$ has already been considered. Let $(\mathcal{G}_t)$ be the "smallest" filtration which contains the initial filtration $(\mathcal{F}_t)$ and such that the random variable A become $\mathcal{G}_0$-measurable. The *theory of enlargement of filtration* gives conditions on A so that all the $(\mathcal{F}_t)$-semimartingales are also $(\mathcal{G}_t)$-semimartingales. In that case, the integral $\int_0^t \phi(A, s)dX_s$ is defined in the usual way by Itô stochastic calculus. The (initial) enlargement of filtrations cannot however always be used for defining such integrals. Even when X is a Brownian motion there are filtrations under which X is no longer a semimartingale, see e.g. Exercise 11.1.

   One central integral in calculus via regularizations is the **forward integral**, which will be introduced below, and for which *substitution formulae* are available. One of them states that, under certain conditions, the forward integral

$\int_0^t \phi(A, s) d^- X_s$ equals the Itô integral $\int_0^t \phi(x, s) dX_s|_{x=A}$, in which the r.v. A substitutes the parameter x. Through those substitution formulae, it is also possible to solve stochastic differential equation with an anticipating initial condition, see [248, 286] and also Sect. 11.3.

Anticipation can also occur considering certain stochastic equations as in (4), but where the positions at time 0 and 1 are given. Equations of the first and second order type were investigated by [251, 250, 249, 85]. Again, in this case, there is no reason why the possible solutions have to be adapted with respect to a Brownian filtration.

Another source of anticipation comes from double integrals. F. Flandoli [116] under the inspiration of A. Chorin (see [56]), proposed several static probabilistic models for vorticity filaments (turbulence, fluidodynamics) based on a standard Brownian motion W. The idea was to find a replacement tool to overcome mathematical problems in Navier-Stokes modeling. The energy associated with these filaments involves apparently innocent expressions of the type $\int_{[0,T]^2} g(W_t - W_s) dW_s dW_t$ for a real function g with a singularity at zero. However previous integral cannot be handled only with basic Itô's calculus techniques. Reasonably, using symmetrization arguments, previous integral should be equal to $2 \int_{[0,T]} \left(\int_0^t g(W_t - W_s) dW_s \right) dW_t$, so to an iterated integral. The inner integral naturally involves W_t as an anticipating random variable: it can be defined via the enlargement of filtrations, or via time reversal tricks as in [116]. If we replace W with a general semimartingale, these tricks cannot be used any more, even when g is smooth. We remark that a precise sense for previous double integrals, when W is replaced by a fractional Brownian motion with index $H > \frac{1}{4}$ and g is singular, was performed in [122]; see also [320] for a related contribution. In [122], the authors used the theory of stochastic currents, see also [121]. The time evolution of a vorticity filament was modeled in [32].

2. **The irregularity.** A particle moving in a random irregular medium can be represented (see [168, 217]) as the "solution" X of a stochastic differential equation of type (4) where the drift $b(t, x) = \beta'(x)$ is the derivative of a two-sided Brownian motion β independent from (W_t), i.e., a Gaussian white noise. Here, the difficulty comes from giving a sense to the formal integral $\int_0^t b(X_s, s) ds = \int_0^t \beta'(X_s) ds$, rigorously characterized for instance in [119, 120, 283, 302], even though the paths $t \mapsto \int_0^t \beta'(X_s) ds$ have no bounded variation. Indeed X_t is the sum of a local martingale and the former integral, and generally it is not a semimartingale. In [119, 120, 283], the authors used the stochastic calculus via regularization and in [302] the theory of Dirichlet forms was exploited.
3. **Gaussian noises.** The class of Gaussian processes is extremely rich in non-semimartingales, even though they constitute a pillar of the theory of stochastic

processes. A famous example is the *fractional Brownian motion* (fbm) (B_t^H), which was introduced by Mandelbrot and Van Ness [215], after a seminal idea of Kolmogorov [186]. H is a parameter called the *Hurst exponent* and belongs to $]0, 1]$. This terminology is related to the contributions of H.E. Hurst and coauthors, see e.g. [169, 170]. A fractional Brownian motion with the Hurst exponent H is a semimartingale if and only if $H = 1/2$. In that case it coincides with the classical Brownian motion. When $H > 1/2$, fractional Brownian motion can be used to model long memory phenomena. It has been frequently used in mathematical finance, the life sciences, hydrology and image recognition, see [255, 157, 39, 54, 331]. Plenty of Gaussian processes have been proposed to generalize fractional Brownian motion, among them we can mention bifractional Brownian motion [166, 284] and multifractal processes, see [10, 11].

The theory of stochastic integration with respect to fractional Brownian motion is relatively recent, see for instance, see [338, 75, 74, 92]. The first paper made use of fractional calculus techniques, the others Malliavin-Skorohod techniques. The literature of the last twenty years includes a huge number of contributions to stochastic calculus with respect to fbm using Skorohod integrals. Typical examples are [6, 5, 7, 228, 190, 189], the latter making use of the stochastic via regularizations. Fractional integral techniques belong to pathwise integration tools, i.e., for every random realization ω, one defines a *deterministic* integral, alternatively to Young type integrals, see considerations below.

Generalized stochastic integration is a way of defining stochastic integrals of the type $\int_0^t Y_s dX_s$ for integrators (X_t) and associated integrands (Y_t).

There are essentially three approaches to define generalized stochastic integrals: pathwise integrals (with its extension constituted by rough paths theory) and Malliavin-Skorohod calculus.

Pathwise integrals One first defines a deterministic "integral" Λ on $\mathcal{C}_1 \times \mathcal{C}_2$, where $\mathcal{C}_i, i = 1, 2$, are two classes of functions defined on $[0, T]$ and $\mathcal{C}_2$ including $C^1([0, T])$. Λ is a suitable extension of $(f, g) \mapsto \int_0^T f(s)dg(s)$ when g is of class C^1. Secondly, the generalized stochastic integral of (Y_t) with respect to (Z_t) is defined as $\Lambda(Y, Z)$, for any processes (Y_t) and (Z_t) such that almost surely, $\big((Y_t)_{t\in[0,T]}, (Z_t)_{t\in[0,T]}\big)$ belongs to $\mathcal{C}_1 \times \mathcal{C}_2$.

One of the most famous pathwise integrals is *Young's.* L.C. Young in 1936 [335] where $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is the class p-variation (resp. q-variation) functions and $\frac{1}{p} + \frac{1}{q} > 1$. It should be recalled that the notion of p-variation was defined by [330], the 1-variation of a function coincides with its total variation and any Hölder continuous function with parameter $\alpha \in]0, 1]$ admits a p-variation for any $p > 1/\alpha$. The corresponding map Λ is then suitable to define generalized stochastic integrals involving the fractional Brownian motion $Z = B^H$ as an integrator and a β-Hölder continuous process (Y_t), with $\beta > 1 - H$ as an as integrand. Indeed the paths of B^H are α-Hölder continuous with $0 < \alpha < H$. The integration approach based on Hölder continuous functions was extended in [278] to Besov spaces using duality between those functional spaces. In a series of papers [338, 340, 339], Zähle introduced generalized integrals via fractional calculus. Other refined (deterministic) integrals of the Riemann type are Kurzweil-Henstock and McShane integrals. The first was introduced independently by Henstock and Kurzweil, see [196, 160, 197, 159]. For the second one we refer for instance to [221]. Some applications to stochastic calculus were considered in [220, 219]. An important step was taken by Föllmer in [125], who, after fixing a family of subdivisions of the interval $[0, T]$, defined a forward type integral $\int_0^T Y dX$ as the limit of $\sum_{i=0}^{N-1} Y_{t_i}(X_{t_{i+1}} - X_{t_i})$ when the mesh of the subdivision converges to zero, $0 = t_0 < \ldots < t_N = T$ being a typical element of the subdivision. It is not, however, our intention to provide a complete discussion on pathwise integrals. A monograph providing a good survey of the history of the subject is [91].

Rough path analysis In [209, 212] T. Lyons and coauthors studied differential equations driven by an integrand process having a p-variation for $p \geq 2$, introducing the so called *rough path theory* which extends in several aspects the techniques of Young's integral, see [335, 336, 337]., The seminal studies of T. Lyons were continued by many authors. Among the contributions, we can cite [204, 155, 133, 167], where in particular were produced deterministic integrals and a related calculus. The theory of differential equations as well as the integrals maintain the calculus (which is completely deterministic) separated from the applications to stochastic analysis.

The basic idea of the rough path theory is that the knowledge of the single path is not sufficient and one needs more information. Indeed, the implementation of the theory does not only depend on an n-dimensional path $x = (x^1, \cdots, x^n)$, but also relies on related multiple integrals. For instance, if x is a p-variation function with $2 \leq p < 3$, the rough path theory applies once the double integrals $\int x^i dx^j$, for any $1 \leq i, j \leq n$ are concretely specified, or equivalently all the Lévy areas $L(x^i, x^j)$, see Sect. 6.7. A significant implementation of this approach to stochastic analysis was performed by [67], where $x = B^H$ is the fractional Brownian motion with the Hurst index H; the authors defined arbitrarily $\int x^i dx^j$ as Skorohod-Stratonovich integrals. Differential equations driven by x admitting Lévy areas are well-posed, if, for instance, the coefficients are smooth and bounded together with their derivatives. Moreover, the solutions depend continuously (with respect to the p-variation distance) on x and on the iterated integrals.

Malliavin-Skorohod calculus Malliavin calculus is an infinite dimensional calculus whose first applications concern the functionals of an underlying Brownian motion. The associated path space is the Banach space $\Omega_0 := C\big([0, T]\big)$ of continuous functions defined over $[0, T]$ and is equipped with the Wiener measure P which is the law of the Brownian motion, considered as random element taking values in Ω_0. One important subspace of the *Wiener space* $C\big([0, T]\big)$ is the *Cameron-Martin* space $W^{1,2}([0, T])$ whose elements are absolutely continuous functions whose derivative belongs to $L^2\big([0, T]\big)$. The first denomination of

this calculus over Ω_0 given by P. Malliavin himself was *stochastic calculus of variations*. We mention four complete books on that topic: [246, 214, 328, 324]. Malliavin calculus has two main classes of applications: the density estimate of a generic random variable and the definition of a stochastic integral with respect to anticipative integrands. In this monograph, we are particularly interested in the second application: Chap. 10 is devoted to a short introduction to the Malliavin calculus techniques applied to stochastic integration. In fact the Malliavin calculus approach to stochastic integration is is often associated to a theory of distributions on Wiener spaces which makes use the so called Sobolev–Watanabe space, see in particular [328]. We mention two alternative methods to this infinite dimensional calculus. One is the so called *white noise calculus*, see [162]; started by Hida. The second one was realized by Krée and his group, see e.g. [188].
The main tool of Malliavin calculus is the *derivative* or the *gradient*. . The Malliavin derivative $\big(D_t F, 0 \le t \le T\big)$ of a random variable F is a square integrable process, i.e., an element of $L^2\big(\Omega \times [0, T]\big)$. The dual map of the Malliavin gradient is the *divergence* operator, also called *Skorohod integral*. It permits defining a generalized stochastic integral $\int_0^T Y \delta W$ for a large class of processes (Y_t). When (Y_t) is a progressively measurable process, then the associated Skorohod integral coincides with the usual Itô integral. Many non-adapted processes are Skorohod integrable, in particular those processes which admit a "good" Malliavin derivative. Malliavin calculus associated with Brownian motion extends to the case of general Gaussian processes replacing the classical Wiener space with the one of abstract Wiener spaces, see [328]. That approach is nevertheless too abstract and not very readable for stochastic calculus purposes. Concrete applications to the calculus related to fractional Brownian motion appeared in [75] and later in [6, 5, 68, 36]. More generally the calculus was extended to general Gaussian processes X represented via an underlying Brownian motion W and several kernel G, see e.g. [7, 247, 206, 228]. Later, an intrinsic Skorohod calculus related to Gaussian processes with given covariance function R was developed in [190, 189]: [190] discussed (resp. [189]) the case when the covariance function R is more regular (resp. more singular) than the covariance of Brownian motion.

Malliavin calculus related to Lévy processes (as well as some generalizations to Lévy measures) has been developed by many authors, see in particular the monograph [83] and references therein.

In the literature appear other definitions of the stochastic integrals, by means of functional analysis type approximations. A classical one is the so called *Ogawa integral*, see [254], which was shown, under certain conditions, to equal a Stratonovich-symmetric integral, see [253].

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About the Book

We now introduce the heart of this monograph: the *stochastic calculus via regularizations*, which is an alternative and unifying method for bringing up stochastic integrals with respect to general integrators and integrands. The foundations of stochastic calculus via regularizations for continuous integrators were settled in [285, 286, 289, 292, 290, 118, 117]: a not so recent survey paper appeared in [293]. In [341], the author considered a useful extension of the forward integral based on a supplementary regularization. In the case where the integrator has jumps, the related theory was first made known in [287] but later, important contributions were made by the Norwegian school, see Chapter 15 of [83] and references therein. More recently a systematic calculus for calculus via regularization with cadlag integrators has been developed in [13]. In the last three decades, calculus via regularization was applied in several circumstances, see, e.g., [205, 82, 35], and even for the stochastic evolution equation, see, e.g., [320]. Regarding financial applications, stochastic calculus via regularization has been successfully used in several cases, which go beyond the enlargement of filtrations, see for instance [34, 207, 83, 71]. Now, in order to exclude arbitrages, in general the underlying stock price has to be a semimartingale, see e.g. [76]. Nevertheless, financial models based on non-semimartingales processes were considered and justified in the regularization framework, see [69] and with other techniques, see e.g. [52, 22, 299, 306, 342]. This approach is related to the first attempt to treat *robustness* in mathematical finance. Other financial applications, this time related to the maximization of the utility of an insider, were also explored in the regularization framework, see e.g. [207, 185]. As mentioned earlier, a significant application to the modeling of vorticity filaments via calculus via regularizations was discussed in [122].

Let us briefly explain the principle of our approach to one-dimensional processes. For the sake of simplicity, we restrict ourselves to a continuous integrator X and a locally integrable integrand Y. In order to define the stochastic integral $\int_0^t Y_s dX_s$ we proceed by first replacing the infinitesimal increment $dX_s = X_{s+ds} - X_s$ by $\frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds$, and second by taking the limit for $\varepsilon \to 0$. Consequently, the *forward* *integral* $\int_0^t Yd^-X$ is defined as the limit in probability as ε goes to 0 of

$$I^-(\varepsilon, Y, dX)(t) := \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds, \quad t \in [0, T], \tag{1}$$

when it exists. We define similarly the *backward integral* (resp. *symmetric integral*) $\int_0^t Yd^+X$ (resp. $\int_0^t Yd^\circ X$) as the limit of $I^+(\varepsilon, Y, dX)(t)$ (resp. $I^\circ(\varepsilon, Y, dX)(t)$) where the elementary increment of X is $\frac{X_s - X_{(s-\varepsilon)+}}{\varepsilon} ds$ $\big($resp. $\frac{X_{s+\varepsilon} - X_{(s-\varepsilon)+}}{2\varepsilon} ds\big)$ instead of $\frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds$.

In the deterministic environment, for instance when either Y or X is a bounded variation continuous function, then taking either the forward or the backward or the symmetric definition does not change the limit. However, in the stochastic setting, the three integrals can be very different. For instance, if Y and X are continuous semimartingales then the forward integral coincides with the usual Itô integral and the symmetric integral equals the Fisk-Stratonovich one.

In the perspective of a stochastic calculus associated with these generalized integrals we introduce the covariation $[X, Y]$ of X and Y as the limit (again in probability) of

$$C(\varepsilon, X, Y)(t) := \int_0^t \frac{(X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)}{\varepsilon} ds, \tag{2}$$

as $\varepsilon \to 0$.

In the particular case where $X = Y$ then $[X] := [X, X]$ is the quadratic variation of X.

Stochastic calculus via regularizations requires a relatively simple formalism. On the one hand it is close to pathwise calculus, on the other hand it takes into account the randomness, being the limits of the ε-integrals in the sense of the convergence in probability. Moreover, it allows connecting different types of pathwise and non pathwise integrals such as Young's, fractional, Skorohod, stemming from the enlargement of filtration and so on.

Let us give briefly a few features of these integrals and covariations.

1. Several algebraic relations exist between these objects. For instance, the symmetric integral is the half-sum of the forward and the backward integrals if they exist.
2. If Y is a progressively measurable process with left limits and X is a continuous semimartingale, then the forward integral $\int_0^t Yd^-X$ equals the Itô integral $\int_0^t YdX$.

3. If Y and X are two continuous semimartingales, then $[Y, X]$ is equal to the usual covariation of the martingale parts of Y and X, denoted by $\langle Y, X\rangle$. In particular, if Y is a square integrable martingale, then $Y_t^2 - [Y, Y]_t$ is a martingale.
4. In the case where $[X]$ exists, X is called *finite quadratic variation process*. A process X for which $[X] = 0$ is called *zero quadratic variation process*.
5. Although $[X, Y]$ is a bounded variation process when X and Y are semimartingales, in the general case this property may fail.
6. When $X = W$ is a Brownian motion and the integrands are adapted, Skorohod and forward integrals equal the Itô integral. However, when the integrand is non-adapted, Skorohod and forward integrals differ by a "trace" term involving the Malliavin derivative of the integrand, see Sect. 10.3 for details.

We come back to the Itô formula (5). Given a smooth function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, with respect to the second variable, one feature of any *generalized stochastic calculus* is to formulate an Itô type formula which expands $f(t, X_t)$, generalizing (5). If $X = M + V$ is a semimartingale, and $f \in C^{1,2}$, an obvious restatement of (5) is

$$f(t, X_t) = M_t^f + A_t^f, \quad t \in [0, T], \tag{3}$$

with

$$M_t^f := f(0, X_0) + \int_0^t \partial_x f(s, X_s) dM_s \tag{4}$$

and

$$A_t^f = \int_0^t \partial_s f(s, X_s) ds + \int_0^t \partial_x f(s, X_s) dV_s + \frac{1}{2} \int_0^t \partial_{xx}^2 f(s, X_s) d < X >_s . \tag{5}$$

The decomposition (3) shows that $f(t, X_t)$ is a semimartingale, M^f being the (local) martingale component.

The quadratic variation appears to be one of the fundamental tools of classical stochastic calculus. In the calculus via regularizations, the quadratic variation and more generally the covariation, constitutes a central object as well. When X is a continuous local martingale, $[X]$ could also be defined as the unique non-decreasing and adapted process such that $X^2 - [X]$ is a local martingale. However, $[X]$ has an intrinsic existence and does not depend on the underlying filtration since, by definition, it is the limit of $C(\varepsilon, X, X)$, as $\varepsilon \to 0$, see (2). The family of finite quadratic variation processes is stable via C^1-transformation, in the sense that when $f \in C^1$, and X is a finite quadratic variation process then the same property holds for $f(X)$ is a finite quadratic variation process, see e.g., Remark 6.1. A semimartingale X is always a finite quadratic variation process. Nevertheless $f(X)$ is not necessarily a semimartingale. Indeed, if W is a Brownian motion, $Y = f(W)$

is not always a semimartingale, see Remark 14.4. We provide in Chap. 8 a list of processes admitting a quadratic variation and which are not semimartingales. Among them, we mention first stochastic Skorohod integrals of the type $X_t = \int_0^t a_s \delta W_s, t \in [0, T]$. On the other hand various Gaussian processes are of finite quadratic variation and they will be investigated in Sect. 8.3. One can also generate finite quadratic variation processes which are not semimartingales by substitution formulae. Indeed, let us consider a random field $\big(X(t, y), t \in [0, T], y \in \mathbb{R}\big)$ such that for every y, $(X(\cdot, y))$ is adapted with respect to a given filtration $(\mathcal{F}_t)$ and is a finite quadratic variation process, for instance a semimartingale. Let G be any random variable, note that $X(\cdot, G)$ is generally not adapted. Under certain additional technical assumptions, it will be shown in Sect. 11.3 that $X(\cdot, G)$ is also a finite quadratic variation process and the *substitution formula* holds, i.e., $[X(\cdot, G)](t) = [X(\cdot, y)]_{|y=G}(t)$. As anticipated earlier, forward integrals also fulfill substitution formulae, as it will be shown in Sect. 11.3. This setting will also be applied to study a stochastic differential equation driven by an $(\mathcal{F}_t)$-semimartingale and for which the initial condition $X_0 = G$ which is possibly not $\mathcal{F}_0$-measurable, see Sect. 12.5.

For any finite quadratic variation process (X_t), stochastic calculus via regularizations based on forward integrals includes the Itô formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^-X_s + \frac{1}{2}\int_0^t f''(X_s)d[X, X]_s, \tag{6}$$

where $f : \mathbb{R} \to \mathbb{R}$ is of class C^2. Itô formulae involving either backward or symmetric integrals also exist, see Sect. 6.2. For instance, we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^\circ X_s. \tag{7}$$

Related Itô formulae, but based on Skorohod integral processes, can be found in the literature, see e.g. [248, 21]. Previous framework can be extended in the three following examples:

1. In stochastic calculus, as well as in the theory of Markov processes, the *local time* of a process X plays a consequential role. If X is a semimartingale, its local time can be defined through the *density occupation formula*. In fact, for $t \in [0, T]$, it can be seen that the application $g \mapsto \int_0^t g(X_s)d\langle X\rangle_s$ admits a.s. a density so there is a random field $(t, a) \mapsto L_t(a)$ (called local time of X) such that $\int_0^t g(X_s)d\langle X\rangle_s = \int_{\mathbb{R}} g(a)L_t(a)da$, for every bounded Borel function $g : \mathbb{R} \to \mathbb{R}$. When f is the difference of two convex functions (or equivalently the second derivative of $f : \mathbb{R} \to \mathbb{R}$ is a Radon measure), an application of the local time consists in the so called *Itô-Tanaka formula*, which consists in reexpressing (5), where the term $\int_0^t f''(X_s)d < X >_s$ (resp. $\int_0^t f'(X_s)dX_s$) is

replaced with

$$\frac{1}{2}\int_{\mathbb{R}} L_t(a) f''(da), \tag{8}$$

(resp.

$$\frac{1}{2}\int_{\mathbb{R}} f'_{-}(X_s) dX_s), \tag{9}$$

where f_{-} is the left-derivative of f and $f"(da)$ is the second derivative (as a Radon measure) of f, in the sense of distributions.

When X is a semimartingale and f is only of class C^1, N. Bouleau and M. Yor in [42] obtained the change of variable formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s - \frac{1}{2}\int_{\mathbb{R}} L_t(da) f'(a), \tag{10}$$

and introduced more generally integrals of the type $\int_{\mathbb{R}} g(a) L_t(da)$ for a continuous function g. When f is a difference of two convex functions, we retrieve the classical Itô-Tanaka formula via an integration by parts of the local time integral in (10). In Sect. 14.3, we will show that the latter integral in (10) can be expressed as an integral via regularizations. In fact, in some cases the local time integral in (10) becomes a classical Itô integral: in particular, when X is a standard Brownian motion, its local time $L_t(a)$ is a semimartingale in a. Several generalizations of the Bouleau-Yor formula were performed when f depends both on space and time, see for instance in [127, 139, 17] or [97].

Let X be a semimartingale with canonical decomposition $X = M + V$. Let us assume moreover that X is a reversible semimartingale, i.e., $\left(X_{T-t},\ t \in [0, T]\right)$ is a semimartingale for any T, see Sect. 14.2. Then for $f:\ [0, T] \times \mathbb{R} \to \mathbb{R}$ of class C^1, it was proved in [290] that the decomposition (3) is still valid, where (M_t^f) is again (4), i.e.,

$$M_t^f = f(0, X_0) + \int_0^t \partial_x f(s, X_s) dM_s, \tag{11}$$

and

$$A_t^f = \int_0^t \partial_s f(s, X_s) dV_s + \frac{1}{2}[\partial_x f(\cdot, X), X]_t, \tag{12}$$

see Theorem 14.1. If f is time-homogeneous, another way to express (3) with (11), (12) is the compact formula (7). Applications and extensions to the case when X is either a Brownian motion or a non-degenerate diffusion (which are in particular time-reversible semimartingale), are given for instance [16]. In [120] the authors investigated the case of diffusions with a possibly irregular drift coefficient and where the function f is time-homogeneous and belongs $W^{1,2}_{loc}$. Extensions to the case where the integrator X has jumps were first obtained by [108], using discretization techniques and recently in [13] using stochastic calculus via regularizations. Other authors have obtained similar expressions to (11) and (12) expressing the covariation term in (12), via a local space-time representation, directly generalizing the Bouleau-Yor formula (10), see e.g. [139, 96] and [98].

The case where the integrator X is multidimensional is indeed more delicate. To our knowledge [290] was the first work concerning the Itô formulae of C^1 functions of semimartingales. When f is a function in $W^{1,2}_{loc}(\mathbb{R}^n)$ and $X = W + x$ where W is an n-dimensional Brownian motion, a formula of the type (13) holds only when x does not belong to a polar set of the Brownian motion, see [128]. If $f \in W^{1,p}_{loc}(\mathbb{R}^n)$ with $p > 2$, generalizing [230], the authors in [231] established that (3) holds for any x in $\mathbb{R}^n$. Other significant contributions come from [267] and [268], in which the remainder process A^f is expressed in terms of local time on curves and surfaces.

2. *Föllmer-Dirichlet processes*, or simply *Dirichlet processes* constitute a natural class which generalize semimartingales. They were introduced in [126] in a discretization framework. A Dirichlet process (X_t) is the sum of a local martingale (M_t) plus a zero quadratic variation process (A_t), where M and A are adapted with respect to a given filtration $(\mathcal{F}_t)$. The decomposition $X = M + A$ is often called *Fukushima decomposition* in the spirit of [135] and is unique once we take $A_0 = 0$. It is clear that a semimartingale is a Dirichlet process since a bounded variation process has zero quadratic variation. Moreover, it is not difficult to show that a Dirichlet process is a finite quadratic variation process. Consequently, the formula (6) holds if $f \in C^2$. On the other hand, if X is a Dirichlet process and f is assumed to be only of class C^1, one may ask if formula (7) holds true. It is not easy to answer this question in full generality, although we have already mentioned that it holds if for instance X is a reversible semimartingale. Exploring another formulation, it can be proved, see Proposition 14.2, that Dirichlet processes are stable under C^1-transformations, namely if (X_t) is a Dirichlet process then $f(X_t)$ remains a Dirichlet process for any function f of class C^1. Moreover

$$f(X_t) = M^f_t + A^f_t, \tag{13}$$

where M^f is defined by (4), i.e., $M^f_t = f(X_0) + \int_0^t f'(X_s)dM_s$, M is the martingale part of X and the quadratic variation of A^f vanishes. Obviously, (13) is the Fukushima decomposition of $f(X_t)$. We remark that Dirichlet processes appear naturally since the image of a semimartingale (for instance a Brownian motion) by a function of class C^1 is a Dirichlet process. The literature contains many significant examples of Dirichlet processes, see e.g. solutions of stochastic differential equation with generalized drift (see [119]) or the total amount of time that a Brownian motion B spends, before some time t, below the level x, composed with B, see [275].

Weak Dirichlet processes generalize Dirichlet processes and were introduced in [107, 149] in the continuous case. A weak Dirichlet process X is the sum of a local martingale M and a *martingale orthogonal process* A, in the sense that A verifies the property $[N, A] = 0$ for any continuous local martingale N. The decomposition $X = M + A$ is unique. Of course, a Dirichlet process is a weak Dirichlet process because the covariation between any local martingale and a zero quadratic variation process A vanishes. Later, in [61], the authors extended this notion to the case of jump processes and [13] formulated a corresponding calculus. We emphasize that the notion of Dirichlet process does not look suitable for the case of cadlag jump process. Indeed if $[A] = 0$ then A is forced to be continuous; in fact $[A]_t \geq \sum_{s \leq t} (\Delta A_s)^2$, see for instance (1.16) in [289]. Moreover, when X is a weak Dirichlet process with finite quadratic variation, (3) and (4) hold with f of class $C^{0,1}$; in that case A^f is a martingale orthogonal process, see Proposition 15.3. In some sense (3) and (4) are a proxy of generalized Itô formula. For instance, those were used in [148] for establishing a verification theorem in stochastic control. Other similar Fukushima decompositions can be found in [307, 322], where one made use of the theory of time-dependent Dirichlet forms. Chapters 14 and 15 are particularly devoted to these important processes.

3. Stochastic calculus via regularization has also been applied with integrators which have no quadratic variation, see [107, 151, 152], and with the slightly different language based on discretization approximations, by [237, 47]. Let us give two significant examples in this direction. The first one concerns processes X which have a finite strong cubic variation, see Sect. 16.3 for details. It can be proved that, whenever $f : \mathbb{R} \to \mathbb{R}$ is of class C^3, the associated Itô formula takes the particular form

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X_s - \frac{1}{12} \int_0^t f^{(3)}(X_s) d[X, X, X]_s \qquad (14)$$

and $[X, X, X]$ denotes the cubic variation of X.

The second example deals with processes having a fourth variation. In that case, we have the Itô formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X_s - \frac{1}{12} \int_0^t f^{(3)}(X_s) d^{\circ(3)} X_s, \qquad (15)$$

where f is of class C^4 and the integral $\int_0^t f^{(3)}(X_s)d^{\circ(3)}X_s$ is defined as the limit in probability of $\int_0^t \frac{f(X_s)+f(X_{s+\varepsilon})}{2}\frac{(X_{s+\varepsilon}-X_s)^3}{\varepsilon}ds$ as $\varepsilon \to 0$. Further developments are provided in Chap. 16. Stochastic differential equations driven by finite cubic variation processes and martingales were investigated in [70]. Those different extensions of Itô's formula illustrate how the calculus via regularization can mix pathwise and probabilistic arguments. These techniques constitute an alternative route to the original pure deterministic rough paths theory.

At the moment of publication of this book, the research activity in calculus via regularizations is still very active, and much attention is conferred to integration and calculus in Banach spaces, see [80, 144, 145, 79, 109, 62, 63, 66, 64]. Operator valued forward integrals were successfully implemented by [272]. Similarly, the fractional calculus type integration of M. Zähle was extended to infinite dimension, see [165].

This book is destined to graduate students in probability and analysis together with mathematicians who are not necessarily specialists in these fields. It develops the theory of stochastic calculus via regularizations as a natural extension of the classical stochastic calculus. Let us briefly present the organization of the content. The first chapter recalls the basic tools in probability. The book is not completely self-contained because for some specific well-known topics (such as the construction of Brownian motion and Doob-Meyer decomposition of a submartingale) we only state the results without any proof: in this case, precise references are mentioned. Generalities related to processes, the definition, and the main properties of semimartingales as well as the definition of Brownian motion are provided in Chap. 2. The reader can find in Chap. 3 definitions and properties of fractional Brownian motion and its extensions. The construction of Itô's stochastic integrals with respect to semimartingales is developed in Chap. 5. In Chap. 6, we recall the usual rules of classical stochastic calculus for semimartingales and more generally those concerning processes having a quadratic variation. A wide variety of such processes is studied in Chap. 8. A specific section, i.e., Chap. 7, is devoted to the change of probability measure on a given probability space. The theory of Itô's stochastic differential equations and its extension to the case where the coefficients are non-Lipschtiz can be found in Chaps. 12 and 13, respectively. As for standard stochastic analysis, we recall the key results concerning Malliavin calculus in Chap. 10. Properties are given that are related to iterated integrals and Hermite polynomials in the companion Chap. 9.

The main tools of stochastic calculus via regularization such as forward, backward, symmetric integrals, covariation, and related properties are presented in Chap. 4. In Sects. 4.4, 4.5, and 4.6, we introduce Young's and fractional integrals and we link them to the integrals via regularization. The stochastic integration based on the theory of enlargement of filtrations and the Malliavin calculus are developed in Chap. 11. We also compare these integrals with the forward/backward/symmetric integrals and in particular integrals obtained by substitution theorems. Chapter 14 is entirely devoted to Dirichlet processes. Chapter 15 focuses on weak Dirichlet processes.

Chapter 16 implements stochastic calculus via regularization in the case of processes only being of finite n-variation for some $n > 2$. A more sophisticated analysis based on compensation and weighted ε-integrals is developed to take into account very irregular integrators. We conclude this book with Chap. 17, which compares stochastic calculus via regularizations and rough paths and in particular weak Dirichlet processes with stochastically controlled processes, i.e., the stochastic version of weakly controlled path (in the sense of Gubinelli), see [155].

Contents

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Francesco Russo studied at EPFL Lausanne and obtained his PhD in Markov random fields. Then, he spent several postdocs in Bielefeld, Bonn, and ENST Paris (now Telecom Paris). Since then, he has been active in various subfields of stochastic analysis with some interests in applications to mathematical physics, mathematical finance, and energy management. He has co-organized many conferences in stochastic analysis and in particular the so called "Ascona Conference," which has been quite influential. He was co-editor of eight volumes of proceedings of that conference, published by Birkäuser. Francesco has had regular collaborations with many international institutions such as the University of Bielefeld, EPFL Lausanne, the University of Campinas (Brazil), and Luiss University in Rome. He was a professor at the University of Paris 13 for almost 15 years where he directed the probability and statistics research team and contributed to the development of probability, and he spent 2 years at the research institution Inria Rocquencourt and in Ecole des Ponts ParisTech.

Pierre Vallois started his research career at Laboratoire de Probabilités in Paris VI. He then held a professorship at University Henri Poincaré (now the University of Lorraine) and carried out his research at Institut Elie Cartan de Lorraine, where he had several responsibilities. Pierre was head of the probability and statistics team, contributing significantly to its development, organizing six probability meetings. He was in charge of the Department of Mathematics in the Faculty of Science and Technology. Pierre was director of the Charles Hermite Federation, which promotes multidisciplinary collaborations between mathematics, computer science, and automation, organizing three forums with industrialists. Since 2018, he is professor emeritus. His research topics are various: Brownian motion, Lévy and diffusion processes, generalized stochastic calculus, and Brownian penalization. Since 2005, he has turned to applications and probabilistic and statistical modeling: tumor growth, biological sequence analysis (DNA), health (allergy), gene networks, and insurance.