

On the well-posedness of a class of McKean Feynman-Kac equations

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Abstract

We analyze the well-posedness of a so called McKean Feynman-Kac Equation (MFKE), which is a McKean type equation with a Feynman-Kac perturbation. We provide in particular weak and strong existence conditions as well as pathwise uniqueness conditions without strong regularity assumptions on the coefficients. One major tool to establish this result is a representation theorem relating the solutions of MFKE to the solutions of a nonconservative semilinear parabolic Partial Differential Equation (PDE).

Key words and phrases. McKean Stochastic Differential Equations; Semilinear Partial Differential Equations; McKean Feynman-Kac equation; Probabilistic representation of PDEs.

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1 Introduction

We discuss in this paper a mean-field type equation of the form

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s) dW_s + \int_0^t [b_0(s, Y_s) + b(s, Y_s, u(s, Y_s))] ds & \forall t \in [0, T], \\ Y_0 \sim \mathbf{u}_0, \text{ a Borel probability measure on } \mathbb{R}^d \\ \int_{\mathbb{R}^d} \varphi(x) u(t, x) dx = \mathbb{E} \left[\varphi(Y_t) \exp \left(\int_0^t \Lambda(s, Y_s, u(s, Y_s)) ds \right) \right] & \forall \varphi \in C_b(\mathbb{R}^d), \forall t \in [0, T], \end{cases} \quad (1)$$

which we will call *McKean-Feynman-Kac Equation* (MFKE). A solution is given by a couple $(Y_s, u(s, \cdot))_{0 \leq s \leq T}$. We refer to the first line of (1) as *Stochastic Differential Equation (SDE)* and to the third line as *linking equation*. The denomination McKean is due to the dependence of the drift coefficient in the SDE not only on time and the position of the process Y but also on

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a function u which, via the linking equation, is related to the distribution of the process. In particular, when $\Lambda = 0$ in the above equation, $u(t, \cdot)$ coincides with the density of the marginal distribution $\mathcal{L}(Y_t)$. In this case, MFKE reduces to a McKean Stochastic Differential Equation (MSDE), which is in general an SDE whose coefficients depend, not only on time and position but also on $\mathcal{L}(Y_t)$. However, in this paper, we emphasize that the drift exhibits a *pointwise* dependence on u ; in particular this dependence is not continuous with respect to the Wasserstein metric, as opposed to the *traditional setting* considered in most of the contributions in the literature. Moreover the drift b is possibly irregular in x .

An interesting feature of MSDEs is that the law of the process Y can often be characterized as the limiting empirical distribution of a large number of interacting particles, whose dynamics are described by a coupled system of classical SDEs. When the number of particles grows to infinity, the given particles behave closely to a system of independent copies of Y . This constitutes the so called *propagation of chaos* phenomenon, already observed in the literature for the case of Lipschitz dependence, with respect to the Wasserstein metric, see e.g. [15, 23, 24, 29, 25].

A second important property of many MSDEs is their close relation to nonlinear PDEs. In the present paper we propose to relate MFKE (1) to a semilinear PDE of the form

$$\begin{cases} \partial_t u = L_t^* u - \operatorname{div}(b(t, x, u)u) + \Lambda(t, x, u)u, & \text{for any } t \in [0, T], \\ u(0, dx) = \mathbf{u}_0(dx), \end{cases} \quad (2)$$

where \mathbf{u}_0 is a Borel probability measure and L^* is the non-degenerate second-order linear partial differential operator such that $a = \Phi\Phi'$ and for all $t \in [0, T]$ and all $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$L_t^*(\varphi)(x) = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 (a_{ij}(t, \cdot)\varphi(\cdot))(x) - \sum_{j=1}^d \partial_j (b_{0,j}(t, \cdot)\varphi(\cdot))(x). \quad (3)$$

When $\Lambda = 0$, the above equation is a *non-linear* Fokker-Planck equation, it is conservative and it is known that, under mild assumptions, it describes the dynamics of the marginal probability densities, $u(t, \cdot)$, of the process Y . This correspondence, between the marginal laws of a diffusion process and a Fokker-Planck type PDE, constitutes a representation property according to which a stochastic object characterizes a deterministic one and vice versa. Such representation results have extensive interesting applications. In physics, biology or economics, it is a way to relate a microscopic model involving interacting particles to a macroscopic model involving the dynamics of some representative quantities. Numerically, this correspondence motivates Monte Carlo approximation schemes for PDEs. In particular, [9] has contributed to develop stochastic particle methods in the spirit of McKean to provide original numerical schemes approaching a PDE related to Burgers equation providing also the rate of convergence.

The idea of generalizing MSDEs to MFKEs was originally introduced in the sequence of papers [20, 19, 21], with an earlier contribution in [3], where $\Lambda(t, x, u) = \xi_t(x)$, ξ being the sample of a Gaussian noise random field, white in time and regular in space. [20] and [19] studied a mollified version of (2), whose probabilistic representation falls into the Wasserstein continuous traditional setting mentioned above. The underlying motivation consisted precisely in

extending, to fairly general non-conservative PDEs, the probabilistic representation of nonlinear Fokker-Planck equations which appears when $\Lambda = 0$. An interesting aspect of this strategy is that it is potentially able to represent an extended class of second order nonlinear PDEs. Allowing $\Lambda \neq 0$ encompasses the case of Burgers-Huxley or Burgers-Fisher equations which are of great importance to represent nonlinear phenomena in various fields such as biology [1, 26], physiology [16] and physics [32]. These equations have the particular interest to describe the interaction between the reaction mechanisms, convection effect, and diffusion transport.

To highlight the contribution of this paper, it is important to consider carefully the two major features differentiating the MFKE (1) from the traditional setting of MSDEs.

To recover the traditional setting one has to do the following.

1. First, one has to put $\Lambda = 0$ in the third line equation of (1). Then $u(t, \cdot)$ is explicitly given by the third line equation of (1) and reduces to the density of the marginal distribution, $\mathcal{L}(Y_t)$. When $\Lambda \neq 0$, the relation between $u(t, \cdot)$ and the process Y is more complex. Indeed, not only does Λ embed an additional nonlinearity with respect to u , but it also involves the whole past trajectory $(Y_s)_{0 \leq s \leq t}$ of the process Y .
2. Secondly, one has to replace the pointwise dependence $b(s, Y_s, u(s, Y_s))$ in equation (1) with a mollified dependence $b(s, Y_s, \int_{\mathbb{R}^d} K(Y_s, y)u(s, y)dy)$, where the dependence with respect to $u(s, \cdot)$ is Wasserstein continuous.

Technically, in the case $\Lambda = 0$, to prove well-posedness of (1) in the traditional setting, one may rely on a fixed point argument in the space of trajectories under the Wasserstein metric. Following the spirit of [29], this approach was carried out in the general case in which Φ also shows a Wasserstein continuous dependence on u in [20]. As already mentioned, the case where the coefficients depend pointwisely on u is far more singular since the dependence of the coefficients on the law of Y is no more continuous with respect to the Wasserstein metrics. In this context, well-posedness results rely on analytical methods and require in general specific smoothness assumptions on both the coefficients and the initial condition. One important contribution in this direction is reported in [14], where strong existence and pathwise uniqueness are established when the diffusion coefficient Φ and the drift $g = b + b_0$ exhibit pointwise dependence on u but are assumed to satisfy strong smoothness assumptions. In this case, the solution u is a classical solution of the PDE. The specific case where the drift vanishes and the diffusion coefficient $\Phi(u(t, Y_t))$ has a pointwise dependence on the law density $u(t, \cdot)$ of Y_t has been more particularly studied in [6] for classical porous media type equations and [7, 2, 5, 4] who obtain well-posedness results for measurable and possibly singular functions Φ . In that case the solution u of the associated PDE (2), is understood in the sense of distributions.

Our analysis of the well-posedness of (1) is based on a different approach. We rely on the notion of mild solutions to Partial Differential Equations (PDEs) involving a reference semigroup on which the solution is built. This is here possible since Φ does not depend on u , so we can allow less regularity on the drift $g = b + b_0$ at least with respect to time and space. To the best of our knowledge, the first attempt to use this approach to analyze McKean type SDEs well-posedness is reported in [22]. The authors considered the case of a classical SDE ($b = 0$), with a more singular dependence of Λ with respect to u , since Λ could also depend on ∇u . In

the present paper, due to the presence of $\operatorname{div}(b(t, y, u)u)$ in the PDE, the semi-group approach needs to be adapted. An integration by parts technique takes advantage of the regularity of the semi-group while allowing to relax the regularity assumptions on b . Consequently, we establish the well-posedness of (1), when Λ and b are only required to be bounded measurable in time and in space and Lipschitz with respect to the third variable. We introduce in particular the cases where strong (resp.) weak solutions appear.

The paper is organized as follows. After this introduction, we clarify in Section 2 the notations and assumptions under which we work and the basic notions of weak and mild solutions of (2) in Section 2.3. In Section 3, we state the main results with related proofs. Theorem 12 provides the equivalence between solutions of the MFKE (1) and the PDE (2). Theorem 13 provides sufficient conditions for existence and uniqueness in law, as well as strong existence and pathwise uniqueness, for the MFKE (1). The rest of the paper is devoted to more technical results used to prove Theorems 12 and 13. In Section 4 we show the equivalence between the notions of weak and mild solutions for (2). Theorem 22 is the key result of Section 5 and states existence and uniqueness of mild solutions for (2). Finally Proposition 23 in Section 6 concerns the uniqueness of the measure-mild solution of the linear PDE (26). It is indeed the crucial tool for proving the existence of a solution to (1).

2 Basic assumptions

2.1 Notations

For a matrix A , A^t denotes its transpose. M_f is the space of signed finite measures on the Borel algebra \mathcal{B}^d of \mathbb{R}^d . We equip M_f with the total variation norm $\|\cdot\|_{TV}$. For $d \in \mathbb{N}^*$ and a function $f = f(t, x, \dots)$ with $t \in [0, T]$ and $x \in \mathbb{R}^d$, we write $\partial_t f = \frac{\partial f}{\partial t}$ and $\partial_{x,k} f := \frac{\partial f}{\partial x_k}$. If there is no ambiguity about x , we sometimes simply write $\partial_k f := \partial_{x,k} f$. Unless we explicitly require regularity of f , we will interpret derivatives as distributional derivatives.

Let E be either \mathbb{R}^d or $[0, T] \times \mathbb{R}^d$, $|\cdot|$ denotes the Euclidean norm of E . $C_0(E)$ is the set of real-valued continuous functions with compact support on E . $C_b(E)$ is the set of real-valued continuous bounded functions on E . $C_0^\infty(E)$ is the set of real-valued smooth functions with compact support on E . For $p \in [1, \infty]$, we write $L^p(E)$ for measurable real-valued functions on E with finite $\|\cdot\|_{L^p}$ norm. and $L_{loc}^p(E)$ for the locally integrable real-valued functions on E . $\mathcal{M}_f(E)$ denotes the space of finite Borel signed measure in E . If $E = \mathbb{R}^d$, we write C_0 instead of $C_0(\mathbb{R}^d)$ and similarly C_b , C_0^∞ and L^p for $C_b(\mathbb{R}^d)$, $C_0^\infty(\mathbb{R}^d)$, and $L^p(\mathbb{R}^d)$ respectively. For $k \in \mathbb{N}^*$, we also set C_b^k to be the set of all bounded functions from \mathbb{R}^d to \mathbb{R} which are bounded and have continuous and bounded partial derivatives up to the k -th order. We then define $C_b^{0,k}([0, T] \times \mathbb{R}^d) := \{f \in C_b([0, T] \times \mathbb{R}^d) \mid f(t, \cdot) \in C_b^k \forall t \in [0, T]\}$.

Finally, we denote the set of symmetric and positive semi-definite matrices in $\mathbb{R}^{d \times d}$ by \mathcal{S}^d .

2.2 Assumptions

In the whole paper we consider a matrix $a = (a_{ij})_{i,j=1}^d : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{S}^d$ such that $a = \Phi\Phi^t$ with $\Phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ as given. We base our discussion on the following assumptions.

A.1 The matrix $a = (a_{ij})_{i,j=1}^d : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{S}^d$ is bounded and measurable. $b_0 : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded and measurable.

A.2 The matrix $a = (a_{ij})_{i,j=1}^d$ is uniformly non-degenerate, i.e. there exists a constant $\mu > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and all $\xi \in \mathbb{R}^d$ we have

$$\xi^t a(t, x) \xi = \sum_{i,j=1}^d a_{ij}(t, x) \xi_i \xi_j \geq \mu |\xi|^2. \quad (4)$$

A.3 For all $x \in \mathbb{R}^d$

$$\lim_{y \rightarrow x} \sup_{0 \leq s \leq T} |a(s, y) - a(s, x)| = 0. \quad (5)$$

Remark 1. *If a is continuous, then (5) is verified.*

These assumptions will come into play in Section 7.2 when we discuss the (weak) existence and uniqueness in law of the SDE in the first line of (1) for a fixed u . This will allow to get weak existence and uniqueness in law for our MFKE. If we substitute assumptions A.3 by the following assumption¹, we will get strong existence and pathwise uniqueness for MFKE.

B.3 Assume that $a : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{S}^d$ is continuous and that $\Phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ which is Lipschitz continuous in space, uniformly in time i.e. there exists a $L_\Phi > 0$ such that for all $t \in [0, T]$ and $x, y \in \mathbb{R}^d$ we have

$$|\Phi(t, x) - \Phi(t, y)| \leq L_\Phi |x - y|.$$

We recall that L_t^* was defined in (3). We define

$$L_t(\varphi)(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \partial_{ij}^2 \varphi(x) + \sum_{j=1}^d b_{0,j}(t, x) \partial_j \varphi(x), \quad (6)$$

for all $t \in [0, T]$ and all $\varphi \in C_0^\infty(\mathbb{R}^d)$. Now we can introduce the Fokker-Planck equation, i.e. for $0 \leq s < T$,

$$\begin{cases} \partial_t \nu(t, x) = L_t^* \nu(t, x) & \forall (t, x) \in (s, T] \times \mathbb{R}^d \\ \nu(s, \cdot) = \nu_0, \end{cases} \quad (7)$$

where ν_0 is a probability measure on \mathcal{B}^d . We introduce the notion of fundamental solution, for simplicity under Assumption A.1.

¹The assumption cited here is a simplified version of a weaker set of assumptions. In fact, we rely here only on the assumptions of Theorem 1 in [30].

Definition 2. A Borel function $p : [0, T] \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a **fundamental solution** to (7), with the convention that $p(s, \cdot, t, \cdot) = 0$ a.e., if $s > t$, if the following holds.

1. For every s, x_0, t such that $0 \leq s < t \leq T$,

$$\int_{\mathbb{R}^d} p(s, x_0, t, x) dx = 1. \quad (8)$$

2. For every probability distribution ν_0 on \mathbb{R}^d , the function

$$\nu_s(t, x) := \int_{\mathbb{R}^d} p(s, x_0, t, x) \nu_0(dx_0) \quad (9)$$

is a solution in the sense of distributions to (7) i.e., for all $\varphi \in C_0^\infty$,

$$\int_{\mathbb{R}^d} \varphi(x) \nu_s(t, x) dx - \int_{\mathbb{R}^d} \varphi(x) \nu_0(dx) = \int_s^t \int_{\mathbb{R}^d} L_r \varphi(x) \nu_s(r, x) dx dr. \quad (10)$$

Remark 3. 1. Proposition 26 provides sufficient conditions for the existence of a fundamental solution.

2. In many examples, a fundamental solution in the sense of Definition 2 is also a fundamental solution of $L_t^* u - \partial_t u = 0$ in the terminology of Friedman, see Definition in sect. 1, p.3 of [13], for instance under the conditions of Proposition 26. The details are provided in the proof.

3. By the validity of (8), the expressions (10) and (9) make sense. We also say that p is a Markov fundamental solution.

Now, we are in the position to impose our assumptions on the fundamental solution.

A.4 There exists a fundamental solution p to (7) with the following properties.

- (a) The first order partial derivatives of the map $x_0 \mapsto p(s, x_0, t, x)$ exist in the distributional sense.
- (b) For almost all $0 \leq s < t \leq T$ and $x_0, x \in \mathbb{R}^d$ there are constants $C_u, c_u > 0$ such that

$$p(s, x_0, t, x) \leq C_u q(s, x_0, t, x) \quad (11)$$

and

$$|\partial_{x_0} p(s, x_0, t, x)| \leq C_u \frac{1}{\sqrt{t-s}} q(s, x_0, t, x), \quad (12)$$

where $q(s, x_0, t, x) := \left(\frac{c_u(t-s)}{\pi}\right)^{\frac{d}{2}} e^{-c_u \frac{|x-x_0|^2}{t-s}}$ is a Gaussian probability density.

- (c) The following *Chapman-Kolmogorov type equality* holds: for all $0 \leq s \leq t \leq r$ and for almost all $x_0, y \in \mathbb{R}^d$ we have

$$p(s, x_0, r, y) = \int_{\mathbb{R}^d} p(s, x_0, t, x) p(t, x, r, y) dx. \quad (13)$$

The next two assumptions concern b and Λ .

A.5 $b : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ is uniformly bounded by $M_b > 0$. Similarly, $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is uniformly bounded by $M_\Lambda > 0$.

A.6 (a) $b : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ is Borel, Lipschitz continuous in its third argument, uniformly in space and time, i.e. there exists an $L_b > 0$ such that for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $z_1, z_2 \in \mathbb{R}$ one has

$$|b(t, x, z_1) - b(t, x, z_2)| \leq L_b |z_1 - z_2|.$$

(b) Similarly, $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is Borel, Lipschitz continuous in its third argument, uniformly in space and time, i.e. there exists an $L_\Lambda > 0$ such that for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $z_1, z_2 \in \mathbb{R}$ one has

$$|\Lambda(t, x, z_1) - \Lambda(t, x, z_2)| \leq L_\Lambda |z_1 - z_2|.$$

(c) We also suppose that $(t, x) \mapsto \Lambda(t, x, 0)$ and $(t, x) \mapsto b(t, x, 0)$ are bounded.

By now, we have imposed assumptions on all the terms appearing in the PDE (2) as well as in MFKE (1). Let us consider now the initial condition of both the PDE and the MFKE.

A.7 \mathbf{u}_0 admits a bounded density u_0 with respect to the Lebesgue measure.

Except for **A.4**, all assumptions are straightforward to verify. Fortunately, there are results specifying manageable conditions which are sufficient for **A.4**.

Remark 4. *Proposition 26 in the Appendix* *precises regularity conditions on the coefficients (a_{ij}) and $(b_0)_j$ which ensure that Assumption A.4 is fulfilled.*

2.3 Weak and mild solutions

We will begin by introducing the notion of a weak solution to (2).

Definition 5. *Assume A.1 and b, Λ to be locally bounded. A **weak solution of PDE (2)** is given by a function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $u \in L^1_{loc}([0, T] \times \mathbb{R}^d)$ and we have, for any $\varphi \in C_0^\infty$,*

$$\int_{\mathbb{R}^d} \varphi(x) u(t, x) dx = \int_{\mathbb{R}^d} \varphi(x) \mathbf{u}_0(dx) + \int_0^t \int_{\mathbb{R}^d} u(s, x) L_s \varphi(x) dx ds$$

$$\begin{aligned}
& + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_j(\varphi(x)) b_j(s, x, u(s, x)) u(s, x) \, dx \, ds \\
& + \int_0^t \int_{\mathbb{R}^d} \varphi(x) \Lambda(s, x, u(s, x)) u(s, x) \, dx \, ds.
\end{aligned} \tag{14}$$

Contrarily to the case when b_0 and b vanish, but Φ may depend on u as in [7, 2, 5], we do not have analytical tools at our disposal with which existence and uniqueness of weak solutions can be established. This is the main reason why we now introduce the notion of mild solutions. Assume that p is a fundamental solution of (7) in the sense of Assumption A.4. The classical and natural formulation of mild solution for (2) looks as follows:

$$\begin{aligned}
u(t, x) &= \int_{\mathbb{R}^d} p(0, x_0, t, x) \mathbf{u}_0(dx_0) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, x_0, u(t, x_0)) u(t, x_0) p(s, x_0, t, x) \, dx_0 \, ds \\
& - \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_{x_0, j} [b_j(t, x_0, u(t, x_0)) u(t, x_0)] p(s, x_0, t, x) \, dx_0 \, ds.
\end{aligned}$$

Note, however, that we do not assume any differentiability of b in space and therefore a definition including $\partial_{x_0, j} b_j(t, x_0, u(t, x_0))$ does not make sense. This motivates to formally integrate by parts. The boundary term would disappear if we set $u \in L^1([0, T] \times \mathbb{R}^d)$ because b is bounded and p satisfies inequality (12). By shifting the derivative from b to p , this leads to the following definition.

Definition 6. *Suppose that assumptions A.1, A.4 and A.5 hold. A function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ will be called **mild solution of (2)** if $u \in L^1([0, T] \times \mathbb{R}^d)$ and we have, for any $t \in [0, T]$, that*

$$\begin{aligned}
u(t, x) &= \int_{\mathbb{R}^d} p(0, x_0, t, x) \mathbf{u}_0(dx_0) + \int_0^t \int_{\mathbb{R}^d} u(s, x_0) \Lambda(s, x_0, u(s, x_0)) p(s, x_0, t, x) \, dx_0 \, ds \\
& + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} u(s, x_0) b_j(s, x_0, u(s, x_0)) \partial_{x_0, j} p(s, x_0, t, x) \, dx_0 \, ds.
\end{aligned} \tag{15}$$

We observe that, whenever $u \in L^1([0, T] \times \mathbb{R}^d)$ and assumptions A.1, A.4 and A.5 hold, the right-hand side of (15) is indeed a well-defined function in $L^1([0, T] \times \mathbb{R}^d)$.

In the sequel we will often make use of the assumption below, which is in fact automatically implied by Assumptions A.1, A.2, A.3.

C The PDE

$$\begin{cases} \partial_t w & = L_t^* w, \\ w(0, \cdot) & = 0, \end{cases} \tag{16}$$

admits $w = 0$ as unique weak solution among measure valued functions from $[0, T]$ to $\mathcal{M}(\mathbb{R}^d)$, i.e. there exists a unique measure-weak solution in the sense of Definition 24.

Remark 7. 1. As we have written above, under Assumptions **A.1**, **A.2** and **A.3**, Assumption **C** is verified. Indeed by Lemma 2.3 in [12], Assumption **C** holds when the martingale problem related to (L_t) is well-posed. Under Assumptions **A.1**, **A.2** and **A.3** this is always the case see e.g. Theorem 7.2.1 in Chapter 7, Section 2 of [28].

2. Other conditions of validity for the uniqueness of weak solutions to (16) are discussed in the literature. Recent results include Theorem 1 in [8], Theorem 1.1 in [27] and Theorem 3.1 in [5].

3 Main results and strategy of the proofs

3.1 Well-posedness for McKean Feynman-Kac equation

Throughout this section, we impose Assumption **A.1**. In particular, **A.1** implies that we can write $a(t, x) = \Phi(t, x)\Phi(t, x)^t$ for some bounded Φ which we fix. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ equipped with a d -dimensional $(\mathcal{F}_t)_{t \in [0, T]}$ Brownian motion $(W_t)_{t \in [0, T]}$ and an \mathcal{F}_0 -measurable random variable $Y_0 \sim \mathbf{u}_0$, we say that a couple (Y, u) is a solution to (1) if the following conditions hold.

1. Y is an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process and $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that (Y, u) verifies (1);
2. $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded.

Below we introduce the notions of existence and uniqueness to (1). The uniqueness aspect will be defined only in the class of pairs (Y, u) such that u is bounded.

Definition 8.

1. We say that (1) admits **strong existence** if for any complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ equipped with a d -dimensional $(\mathcal{F}_t)_{t \in [0, T]}$ Brownian motion $(W_t)_{t \in [0, T]}$ and an \mathcal{F}_0 -measurable random variable $Y_0 \sim \mathbf{u}_0$, there is a couple (Y, u) such that Y is an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process, $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and (Y, u) is a solution to (1).
2. We say that (1) admits **pathwise uniqueness** if for any complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ equipped with a d -dimensional $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion $(W_t)_{t \in [0, T]}$, an \mathcal{F}_0 -random variable $Y_0 \sim \mathbf{u}_0$, the following holds. For any given two pairs (Y^1, u^1) and (Y^2, u^2) of solutions to (1) such that u^1, u^2 are bounded and $Y_0^1 = Y_0^2$ \mathbb{P} -a.s. we have that Y^1 and Y^2 are indistinguishable and $u^1 = u^2$.

Definition 9.

1. We say that (1) admits **existence in law** if there is a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ equipped with a d -dimensional $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion $(W_t)_{t \in [0, T]}$, a pair (Y, u) solution of (1), where Y is an $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process and u is a real valued function defined on $[0, T] \times \mathbb{R}^d$.
2. We say that (1) admits **uniqueness in law**, if the following holds. For any two solutions, (Y, u) and (\tilde{Y}, \tilde{u}) of (1), defined on complete filtered probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{\mathbb{P}})$, respectively, which are such that u and \tilde{u} are bounded, then $u = \tilde{u}$ and Y and \tilde{Y} have the same law.

3.2 A relaxation of Assumption A.5

. While Assumption A.5 is convenient, it is not verified in many interesting situations. However, it can be replaced with the following assumption.

C5 There exists a mild solution $u \in L^1([0, T] \times \mathbb{R}^d)$ of (2), such that $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded.

Remark 10. Under Assumption A.6, if u is a bounded mild solution of (2) then Assumption C5 is verified.

Remark 11. For example, consider Burgers' equation

$$\begin{cases} \partial_t u(t, x) &= \frac{\nu}{2} \partial_{xx}^2 u(t, x) - u(t, x) \partial_x u(t, x), & \forall (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, \cdot) &= u_0, \end{cases} \quad (17)$$

where the constant $\nu > 0$ and u_0 is a bounded probability density with respect to the Lebesgue measure. In our framework, the representation would be $b(t, x, z) = \frac{1}{2}z$, $\Lambda(t, x, z) = 0$, $b_0(t, x) = 0$ and $\Phi(t, x) = \sqrt{\nu}$. As the reader may easily verify using Proposition 26, all assumptions except for A.5 are satisfied. Assumption A.5 is violated because $z \mapsto \frac{1}{2}z$ is only locally bounded. However Assumption C5 is verified. Indeed there exists a bounded classical solution u . In fact for instance [10] states that (17) admits a classical solution given by

$$u(t, x) = \frac{\mathbb{E}[u_0(x + \nu B_t) e^{-\frac{U_0(x + \nu B_t)}{\nu^2}}]}{\mathbb{E}[e^{-\frac{U_0(x + \nu B_t)}{\nu^2}}]}, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad (18)$$

where B denotes a one-dimensional Brownian motion and U_0 is the cumulative distribution function associated with u_0 . Since u_0 is bounded, u is obviously bounded. This is therefore a weak solution. Taking into account Assumption C and Proposition 16, it is also a mild solution. Then by Remark 10, Assumption C5 is verified.

3.3 Main results

In this section, we state the two main theorems of the article.

Theorem 12. Assume that Assumptions A.1, A.2, A.3, A.4 hold.

1. Let (Y, u) be a solution of (1), where $u \in L^1([0, T] \times \mathbb{R}^d)$: in particular $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded. Then u is a weak solution of (2).
2. Let $u \in L^1([0, T] \times \mathbb{R}^d)$ be a weak solution of (2) such that $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded. Then (1) admits existence and uniqueness in law. In particular there is a (unique in law) process Y such that (Y, u) is a solution of (1).
3. If Assumption A.3 is replaced with B.3, we obtain an analogous results to item 2. where existence and uniqueness in law is replaced by strong existence and pathwise uniqueness.

This, together with Theorem 22 will allow us to formulate existence and uniqueness for the MFKE equation.

Theorem 13. Assume that [A.1](#), [A.2](#), [A.3](#), [A.4](#), [A.6](#), [A.7](#) hold.

1. Under Assumption [A.5](#) or Assumption [C5](#) (1) admits existence in law. Under Assumption [A.5](#) the constructed solution (Y, u) is such that u is bounded.
2. (1) admits uniqueness in law.
3. Suppose [A.3](#) is replaced with [B.3](#).
 - Under Assumption [A.5](#) or [C5](#) we have strong existence of a solution to (1).
 - Under Assumption [A.5](#) we have pathwise uniqueness of a solution (Y, u) to (1).

Remark 14. Concerning item 1. of the theorem above, if we substitute [A.3](#) with the hypothesis that Φ is non-degenerate, i.e. (4) is fulfilled replacing a with Φ , we still have weak existence of a solution (Y, u) (but not necessarily uniqueness) to (1). Instead of appealing to Lemma 27 for the weak existence and uniqueness in law of a solution to the SDE, we then appeal to Remark 28.

3.4 Strategy of the proof

In this section, we sketch the proofs to our main theorems, beginning with Theorem 12.

3.4.1 Proof of Item 1. of Theorem 12

Item 1. of Theorem 12 follows from Proposition 15 below, which is obtained by a direct application of Itô's formula.

Proposition 15. Suppose the validity of Assumption [A.1](#). Suppose that (Y, u) is a solution of (1). Then u is a weak solution of (2).

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be a test function. We use stochastic integration by parts to infer that

$$\begin{aligned} \varphi(Y_t) \exp\left(\int_0^t \Lambda(s, Y_s, u(s, Y_s)) ds\right) &= \varphi(Y_0) \exp(0) + \int_0^t \exp\left(\int_0^s \Lambda(r, Y_r, u(r, Y_r)) dr\right) d\varphi(Y_s) \\ &\quad + \int_0^t \varphi(Y_s) d \exp\left(\int_0^s \Lambda(r, Y_r, u(r, Y_r)) dr\right). \end{aligned} \quad (19)$$

To further develop the second term on the right-hand side, an application of Itô's formula yields

$$\begin{aligned} \varphi(Y_s) &= \varphi(Y_0) + \sum_{i=1}^d \int_0^s \partial_i \varphi(Y_r) dY_r^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^s \partial_{ij}^2 \varphi(Y_r) d\langle Y^i, Y^j \rangle_r \\ &= \varphi(Y_0) + \sum_{i=1}^d \left(\sum_{j=1}^d \int_0^s \partial_i \varphi(Y_r) \Phi_{ij}(r, Y_r) dW_r^j + \int_0^s \partial_i \varphi(Y_r) (b_{0,i}(r, Y_r) + b_i(r, Y_r, u(r, Y_r))) dr \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{i,j=1}^d \int_0^s a_{ij}(r, Y_r) \partial_{ij}^2 \varphi(Y_r) dr \\
& = \varphi(Y_0) + \sum_{i,j=1}^d \int_0^s \partial_i \varphi(Y_r) \Phi_{ij}(r, Y_r) dW_r^j + \int_0^s L_r \varphi(Y_r) dr + \sum_{i=1}^d \int_0^s \partial_i \varphi(Y_r) b_i(r, Y_r, u(r, Y_r)) dr.
\end{aligned}$$

For the third term we write

$$\exp\left(\int_0^s \Lambda(r, Y_r, u(r, Y_r)) dr\right) = 1 + \int_0^s \exp\left(\int_0^r \Lambda(z, Y_z, u(z, Y_z)) dz\right) \Lambda(r, Y_r, u(r, Y_r)) dr.$$

Now, plugging all this into equation (19) leaves us with

$$\begin{aligned}
& \varphi(Y_t) \exp\left(\int_0^t \Lambda(s, Y_s, u(s, Y_s)) ds\right) \\
& = \varphi(Y_0) \exp(0) + \sum_{i,j=1}^d \int_0^t \exp\left(\int_0^s \Lambda(r, Y_r, u(r, Y_r)) dr\right) \partial_i \varphi(Y_s) \Phi_{ij}(s, Y_s) dW_s^j \\
& \quad + \sum_{i=1}^d \int_0^t \exp\left(\int_0^s \Lambda(r, Y_r, u(r, Y_r)) dr\right) \partial_i \varphi(Y_s) b_i(s, Y_s, u(s, Y_s)) ds \\
& \quad + \int_0^t \exp\left(\int_0^s \Lambda(r, Y_r, u(r, Y_r)) dr\right) L_s \varphi(Y_s) ds \\
& \quad + \int_0^t \varphi(Y_s) \exp\left(\int_0^s \Lambda(z, Y_z, u(z, Y_z)) dz\right) \Lambda(s, Y_s, u(s, Y_s)) ds,
\end{aligned}$$

which almost finishes the proof. Indeed, we now only have to take the expectation using Fubini's theorem to exchange the integral with respect to time with the expectation and then apply the third line of (1) which gives exactly (14). \square

3.4.2 Sketch of the proof of Item 2. of Theorem 12

To establish item 2. of Theorem 12, we proceed through the following steps.

1. Let u be a (weak) solution to (2) such that $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded.
2. By Proposition 16, u is also a mild solution to (2).
3. Using Stroock-Varadhan arguments, see Lemma 27, we construct a process Y satisfying (in law) the SDE

$$\begin{cases} Y_t & = Y_0 + \int_0^t \Phi(s, Y_s) dW_s + \int_0^t (b_0(s, Y_s) + b(s, Y_s, u(s, Y_s))) ds \quad \forall t \in [0, T], \\ Y_0 & \sim \mathbf{u}_0. \end{cases} \quad (20)$$

4. Let $t \geq 0$. We can then define a measure μ_t on $(\mathbb{R}^d, \mathcal{B}^d)$ by setting for any measurable and bounded $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$

$$\int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) := \mathbb{E} \left[\varphi(Y_t) \exp \left(\int_0^t \Lambda(s, Y_s, u(s, Y_s)) ds \right) \right]. \quad (21)$$

We show that μ_t is a solution in the sense of distributions (i.e. a measure-weak solution in the sense of Definition 24) of the linear equation (26) with $\hat{\Lambda}(s, y) = \Lambda(s, y, u(s, y))$, $\hat{b}(s, y) = b(s, y, u(s, y))$. In fact, we basically apply integration by parts and Itô's formula on $\varphi(Y_t) \exp \left(\int_0^t \Lambda(s, Y_s, u(s, Y_s)) ds \right)$ similarly to the proof of Proposition 15. According to Proposition 25, this implies that μ is a measure-mild solution of (26) in the sense of Definition 18.

5. On the other hand, by item 2. $\nu_t(dx) = u(t, x) dx$ is obviously also a measure-mild solution of (26). We thus have two measure-mild solutions of (26), $\mu_t(dx)$ and $\nu_t(dx)$.
6. Proposition 23 states uniqueness of measure-mild solutions to (26). This allows us to infer $\mu_t(dx) = u(t, x) dx$, thereby completing the proof of item 2. of Theorem 12.

Item 3. can be established replacing Stroock-Varadhan arguments (see Lemma 27) with Lemma 29 which states strong existence of (20).

3.4.3 Sketch of the proof of Theorem 13

Now, we outline the proof of Theorem 13.

1. Let u be a mild solution of (2), as stated by Assumption C5 or guaranteed by item 1. of Theorem 22 if Assumption A.5 holds; in that latter case u is bounded. According to Proposition 16, u is then a weak solution of (2). Hence, by item 2. of Theorem 12, we conclude that there is a stochastic process Y such that the couple (Y, u) solves (1) in law.
2. As for uniqueness, suppose there are two solutions (Y_1, u_1) and (Y_2, u_2) of (1) such that u_1, u_2 are bounded. In that case by Assumption A.6 $(s, x) \mapsto b(s, x, u_i(s, x)), (s, x) \mapsto \Lambda(s, x, u_i(s, x))$ are bounded. Then, by item 1. of Theorem 12, both u_1 and u_2 are weak solutions to (2). Then, by Proposition 16, u_1 and u_2 are also mild solutions to (2). But by item 2. of Theorem 22, mild solutions are unique, i.e. $u_1 = u_2$. We conclude by Lemma 27 that the law of Y_1 is the same as the law of Y_2 .
3. The same proof works substituting item 2. of Theorem 12 with item 3. and Lemma 27 with Lemma 29.

In the sequel of the paper we will state and establish the main ingredients of these proofs.

4 Equivalence of weak and mild solutions of the nonlinear PDE

The main result of this section is Proposition 16, which specifies conditions under which weak and mild solutions to (2) are equivalent.

Proposition 16. *Let u such that $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded. Assume A.1 and A.4.*

1. *If u is a mild solution of (2) then it is also a weak solution.*
2. *Conversely, if Assumption C holds and u is a weak solution of (2), then u is also a mild solution of (2).*

Proposition 16 is a key result for this work because we do not have analytical tools at our disposal to establish existence or even uniqueness of a weak solution directly. However, we can apply fixed-point theorems to prove existence and uniqueness of mild solutions. Proposition 16 states that this mild solution is also a weak solution which proves in particular the existence of weak solutions. Additionally, we can even infer on the uniqueness of weak solutions if we can ensure that (16) admits no weak solution other than 0. For the proof of the Proposition we need a technical lemma.

Lemma 17. *Let us suppose Assumptions A.1 and A.4. Fix an arbitrary function $\tilde{\Lambda} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ in $L^1([0, T] \times \mathbb{R}^d)$ and an arbitrary function $\tilde{b} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ in $L^1([0, T] \times \mathbb{R}^d)$. Define the function $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ by*

$$\begin{aligned} v(t, x) &= \int_{\mathbb{R}^d} p(0, x_0, t, x) \mathbf{u}_0(dx_0) + \int_0^t \int_{\mathbb{R}^d} \tilde{\Lambda}(s, x_0) p(s, x_0, t, x) dx_0 ds \\ &\quad - \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \tilde{b}_j(s, x_0) \partial_{x_0, j} p(s, x_0, t, x) dx_0 ds. \end{aligned} \quad (22)$$

Then for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ and for any $t \in [0, T]$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) v(t, x) dx &= \int_{\mathbb{R}^d} \varphi(x) \mathbf{u}_0(dx) + \int_0^t \int_{\mathbb{R}^d} v(s, x) L_s \varphi(x) dx ds \\ &\quad + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_j(\varphi(x)) \tilde{b}_j(s, x) dx ds + \int_0^t \int_{\mathbb{R}^d} \varphi(x) \tilde{\Lambda}(s, x) dx ds. \end{aligned} \quad (23)$$

Proof. The idea of the proof is to generalize the statement formulated in Step 0., below, by approximating b and Λ with smooth functions. It is thus natural to structure the proof as follows.

Step 0. Fubini and the fundamental solution

Define the function $\tilde{v} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\tilde{v}(s, x) = \int_{\mathbb{R}^d} p(0, x_0, s, x) \mathbf{u}_0(dx_0) + \int_0^s \int_{\mathbb{R}^d} \tilde{\Lambda}(r, x_0) p(r, x_0, s, x) dx_0 dr. \quad (24)$$

Using Fubini's theorem and the definition of the fundamental solution, it is easy to check that \tilde{v} satisfies for any test function $\varphi \in C_0^\infty(\mathbb{R}^d)$ and any $t \in [0, T]$

$$\int_0^t \int_{\mathbb{R}^d} \tilde{v}(s, x) L_s \varphi(x) dx ds = \int_{\mathbb{R}^d} \varphi(x) \tilde{v}(t, x) dx - \int_{\mathbb{R}^d} \varphi(x) \tilde{v}(0, x) dx - \int_0^t \int_{\mathbb{R}^d} \varphi(x) \tilde{\Lambda}(s, x) dx ds.$$

Step 1. Suppose that the mapping $\mathbf{x} \mapsto \tilde{\Lambda}(t, \mathbf{x})$ is bounded and that the map $(t, \mathbf{x}) \mapsto \tilde{\mathbf{b}}_j(t, \mathbf{x})$ belongs to $C_b^{0,1}$ for all $j = 1, \dots, d$.

In this case, (22) can be manipulated by integration by parts to read

$$v(s, x) = \int_{\mathbb{R}^d} p(0, x_0, s, x) \mathbf{u}_0(dx_0) + \int_0^s \int_{\mathbb{R}^d} \left[\tilde{\Lambda}(r, x_0) + \left(\sum_{j=1}^d \partial_{x_{0,j}} \tilde{b}_j(r, x_0) \right) \right] p(r, x_0, s, x) dx_0 dr.$$

Note that the boundary term in the integration by parts vanishes because \tilde{b} is bounded and p satisfies inequality (12). Because \tilde{b} has bounded derivatives, the function, $\Psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that $\Psi(t, x) := \tilde{\Lambda}(t, x) + \sum_{j=1}^d \partial_j \tilde{b}_j(t, x)$ is bounded, too. Thus we can use the statement of Step 0. with $\tilde{\Lambda}(t, x) := \Psi(t, x)$. Thus we immediately obtain the desired equation (23) by inserting the definition of Ψ and using the linearity of the integral and integrating by parts in the reverse sense of above computation.

Step 2. Assume that the maps $(t, \mathbf{x}) \mapsto \tilde{\mathbf{b}}_j(t, \mathbf{x})$ and $(t, \mathbf{x}) \mapsto \tilde{\Lambda}(t, \mathbf{x})$ are in $L^1([0, T] \times \mathbb{R}^d)$ for all $j = 1, \dots, d$.

Because $C_0^\infty([0, T] \times \mathbb{R}^d)$ is dense in $L^1([0, T] \times \mathbb{R}^d)$, there exist sequences $(b_{n,j})_{n \in \mathbb{N}}$, $j = 1, \dots, d$ and $(\Lambda_n)_{n \in \mathbb{N}}$, in $C_0^\infty([0, T] \times \mathbb{R}^d)$, such that $(b_{n,j})_{n \in \mathbb{N}}$ converges to \tilde{b}_j and $(\Lambda_n)_{n \in \mathbb{N}}$ converges to $\tilde{\Lambda}$ with respect to $\|\cdot\|_{L^1([0, T] \times \mathbb{R}^d)}$. For any $n \in \mathbb{N}$, we define the function $v_n : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$v_n(s, x) = \int_{\mathbb{R}^d} p(0, x_0, s, x) \mathbf{u}_0(dx_0) + G_n(s, x) + F_{n,j}(s, x),$$

where

$$G_n(s, x) := \int_0^s \int_{\mathbb{R}^d} \Lambda_n(r, x_0) p(r, x_0, s, x) dx_0 dr$$

$$F_{n,j}(s, x) := \int_0^s \int_{\mathbb{R}^d} b_{n,j}(r, x_0) \partial_{x_0,j} p(r, x_0, s, x) dx_0 dr.$$

We will prove that $F_{n,j}$ converges, as n tends to infinity, in $\|\cdot\|_{L^1([0,T] \times \mathbb{R}^d)}$, to $F_j : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$F_j(s, x) := \int_0^s \int_{\mathbb{R}^d} \tilde{b}_j(r, x_0) \partial_{x_0,j} p(r, x_0, s, x) dx_0 dr.$$

Indeed, by (12) and the fact that $q(r, x_0, s, x)$ is a probability density for almost all r, x_0, s and Tonelli we get

$$\begin{aligned} \|F_{n,j}(s, x) - F_j(s, x)\|_{L^1([0,T] \times \mathbb{R}^d)} &\leq \int_0^T \int_{\mathbb{R}^d} \int_0^s \int_{\mathbb{R}^d} |b_{n,j}(r, x_0) - \tilde{b}_j(r, x_0)| |\partial_{x_0,j} p(r, x_0, s, x)| dx_0 dr dx ds \\ &\leq \int_0^T \int_{\mathbb{R}^d} \int_0^s \int_{\mathbb{R}^d} |b_{n,j}(r, x_0) - \tilde{b}_j(r, x_0)| \frac{C_u}{\sqrt{s-r}} q(r, x_0, s, x) dx_0 dr dx ds \\ &= \int_0^T \int_{\mathbb{R}^d} \int_0^s |b_{n,j}(r, x_0) - \tilde{b}_j(r, x_0)| \frac{C_u}{\sqrt{s-r}} \int_{\mathbb{R}^d} q(r, x_0, s, x) dx dr dx_0 ds \\ &= C_u \int_0^T \left\| b_{n,j}(r, \cdot) - \tilde{b}_j(r, \cdot) \right\|_{L^1} \int_r^T \frac{1}{\sqrt{s-r}} ds dr \\ &\leq 2C_u \sqrt{T} \int_0^T \left\| b_{n,j}(r, \cdot) - \tilde{b}_j(r, \cdot) \right\|_{L^1} dr = 2C_u \sqrt{T} \left\| b_{n,j} - \tilde{b}_j \right\|_{L^1([0,T] \times \mathbb{R}^d)}. \end{aligned}$$

To conclude, we now simply take the limit $n \rightarrow \infty$. With similar arguments, we prove that $G_n[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ converges, as n tends to infinity, in $\|\cdot\|_{L^1([0,T] \times \mathbb{R}^d)}$, to $G : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, where

$$G(s, x) := \int_0^s \int_{\mathbb{R}^d} \tilde{\Lambda}(r, x_0) p(r, x_0, s, x) dx_0 dr.$$

From this, we conclude that v_n converges in $\|\cdot\|_{L^1([0,T] \times \mathbb{R}^d)}$ to v as defined in (22). According to Step 1. of the proof, for any $n \in \mathbb{N}$ and any $\varphi \in C_0^\infty(\mathbb{R}^d)$ we also have

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) v_n(t, x) dx &= \int_{\mathbb{R}^d} \varphi(x) v(0, x) dx + \int_0^t \int_{\mathbb{R}^d} v_n(s, x) L_s \varphi(x) dx ds \\ &\quad + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_j(\varphi(x)) b_{n,j}(s, x) dx ds + \int_0^t \int_{\mathbb{R}^d} \varphi(x) \Lambda_n(s, x) dx ds. \end{aligned}$$

We now take the limit as n tends to infinity on both sides of this equation and only have to justify that we can exchange the limits and the integrals. For this, it suffices to observe that whenever a sequence of functions converges in any L^1 -space, then multiplying this function with a bounded function preserves the L^1 -convergence. \square

Now that we have completed the proof of Lemma 17, we turn to the proof of Proposition 16.

Proof (of Proposition 16).

a) First suppose that u is a mild solution of PDE (2) in the sense of Definition 6.

Then we set $\tilde{\Lambda}(s, x) = \Lambda(s, x, u(s, x))u(s, x)$ and $\tilde{b}(s, x) = b(s, x, u(s, x))u(s, x)$ which are both in $L^1([0, T] \times \mathbb{R}^d)$ because $\Lambda(s, x, u(s, x))$ and $b(s, x, u(s, x))$ are bounded by assumption and u is in $L^1([0, T] \times \mathbb{R}^d)$ according to the definition of the mild solution. Hence we can apply Lemma 17 which directly yields that u is a weak solution.

b) Now suppose that u is a weak solution of PDE (2) in the sense of Definition 5.

Then, we define

$$\begin{aligned} v(t, x) := & \int_{\mathbb{R}^d} p(0, x_0, t, x) \mathbf{u}_0(dx_0) + \int_0^t \int_{\mathbb{R}^d} u(s, x_0) \Lambda(s, x_0, u(s, x_0)) p(s, x_0, t, x) dx_0 ds \\ & + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} u(s, x_0) (b_j(u, x_0, u(s, x_0))) \partial_{x_{0,j}} p(s, x_0, t, x) dx_0 ds. \end{aligned}$$

The strategy of this proof is to establish that $v = u$. Again, we apply Lemma 17 with $\tilde{\Lambda}(s, x) = \Lambda(s, x, u(s, x))u(s, x)$ and $\tilde{b}(s, x) = b(s, x, u(s, x))u(s, x)$ on v and infer that v is a weak solution of the PDE

$$\begin{cases} \partial_t v(t, x) &= L_t^* v(t, x) - \sum_{j=1}^d \partial_j (b_j(t, x, u(t, x))u(t, x)) + u(t, x) \Lambda(t, x, u(t, x)) \\ v(0, \cdot) &= u_0. \end{cases} \quad (25)$$

By assumption, u is a weak solution of PDE (2), so it is in particular a weak solution of (25). Hence $u - v$ is a solution of the PDE (16). But as by Assumption C the unique weak solution of (16) is the constant function equal to zero, we conclude that $v = u$ is a mild solution of (2). \square

5 Existence and uniqueness of a mild solution to the nonlinear PDE

In this section Assumption A.4 is in force. We fix some bounded Borel measurable functions $\hat{\Lambda} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\hat{b} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$.

We begin by introducing the notion of a measure-mild solution of the linearized equation

$$\begin{cases} \partial_t v(t, x) &= L_t^* v(t, x) - \sum_{j=1}^d \partial_j (\hat{b}_j(t, x) v(t, x)) + \hat{\Lambda}(t, x) v(t, x), \\ v(0, \cdot) &= \mathbf{u}_0. \end{cases} \quad (26)$$

In fact (26) constitutes the linearized version of PDE (2).

5.1 Notion of measure-mild solution

Recall that p denotes the fundamental solution to (7) as introduced in Definition 2.

Definition 18. Assume A.4 and that \hat{b} and $\hat{\Lambda}$ are bounded. A measure-valued map $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ will be called **measure-mild solution** of (26) if for all $t \in [0, T]$

$$\begin{aligned} \mu_t(dx) &= \int_{\mathbb{R}^d} p(0, x_0, t, x) dx \mathbf{u}_0(dx_0) + \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) p(r, x_0, t, x) dx \mu_r(dx_0) dr \\ &\quad + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \hat{b}_j(r, x_0) \partial_{x_0, j} p(r, x_0, t, x) dx \mu_r(dx_0) dr. \end{aligned} \quad (27)$$

Remark 19.

1. Note that a measure is characterized by the integrals of all smooth functions with bounded support. This is why the definition of a measure-mild solution in (27) is equivalent to require that for any $\varphi \in C_0^\infty$ and for any $t \in [0, T]$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(x) p(0, x_0, t, x) dx \right) \mathbf{u}_0(dx_0) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) \left(\int_{\mathbb{R}^d} \varphi(x) p(r, x_0, t, x) dx \right) \mu_r(dx_0) dr \\ &\quad + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \hat{b}_j(r, x_0) \left(\int_{\mathbb{R}^d} \varphi(x) \partial_{x_0, j} p(r, x_0, t, x) dx \right) \mu_r(dx_0) dr. \end{aligned}$$

2. If one compares the definition of measure-mild solutions, setting

$$\hat{b}(s, x) = b(s, x, u(s, x)), \quad \hat{\Lambda}(s, x) = \Lambda(s, x, u(s, x)),$$

to the one of mild solutions in Definition 6, it first appears that the only difference being that $u(t, x) dx$ has been replaced by $\mu(t, dx)$. In particular, if u is a mild solution of (2), see Definition 6, then $\mu_t(dx) := u(t, x) dx$ is a measure-mild solution of (26).

In Proposition 23 we will show under Assumption A.4 uniqueness of measure-mild solution of the linear equation (26).

5.2 Proof of Theorem 22

We will adapt the construction of a mild solution given in [22] to the case of PDE (2). We will apply the Banach fixed-point theorem to a suitable mapping in order to prove existence and uniqueness of a mild solution on a *small* time interval.

We will frequently make use of the notation $L^1([t_1, t_2], L^1(\mathbb{R}^d))$ for $t_1 < t_2$, which is the set of functions f on $[t_1, t_2]$ with values in $L^1(\mathbb{R}^d)$ such that $\|f\|_1 < \infty$, where this norm is defined as

$$\|f\|_1 := \int_{t_1}^{t_2} \|f(s)(\cdot)\|_{L^1(\mathbb{R}^d)} ds. \quad (28)$$

Fix $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $r \in [0, T[$ and $\tau \in]0, T - r]$. We define $\widehat{u}_0(r, \phi) : [r, r + \tau] \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\widehat{u}_0(r, \phi)(t, x) := \int_{\mathbb{R}^d} p(r, x_0, t, x) \phi(x_0) dx_0. \quad (29)$$

We will often forget to indicate the r parameter and note $\widehat{u}_0(\phi)$ instead of $\widehat{u}_0(r, \phi)$. By (11), the following bounds hold

$$\|\widehat{u}_0(r, \phi)\|_1 \leq \|\phi\|_1 \quad \text{and} \quad \|\widehat{u}_0(r, \phi)\|_\infty \leq C_u \|\phi\|_\infty. \quad (30)$$

We define $\Pi : L^1([r, r + \tau], L^1(\mathbb{R}^d)) \rightarrow L^1([r, r + \tau], L^1(\mathbb{R}^d))$ by

$$\begin{aligned} \Pi(r, v)(t, x) &:= \int_r^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \widehat{\Lambda}(s, x_0, v + \widehat{u}_0(\phi)) dx_0 ds \\ &\quad + \sum_{j=1}^d \int_r^t \int_{\mathbb{R}^d} \partial_{x_{0,j}} p(s, x_0, t, x) \widehat{b}_j(s, x_0, v + \widehat{u}_0(\phi)) dx_0 ds, \end{aligned} \quad (31)$$

where we have used the shorthand notation

$$\begin{aligned} \widehat{\Lambda}(s, x_0, v + \widehat{u}_0(\phi)) &:= \Lambda(s, x_0, v(s, x_0) + \widehat{u}_0(\phi)(s, x_0))(v(s, x_0) + \widehat{u}_0(\phi)(s, x_0)), \\ \widehat{b}(s, x_0, v + \widehat{u}_0(\phi)) &:= b(s, x_0, v(s, x_0) + \widehat{u}_0(\phi)(s, x_0))(v(s, x_0) + \widehat{u}_0(\phi)(s, x_0)). \end{aligned}$$

We remark that the map Π depends on r, τ but this will be omitted in the sequel. We will denote the centered closed ball of radius M in $L^1(\mathbb{R}^d)$ by $B[0, M]$ and the closed centered ball in $L^\infty([r, r + \tau] \times \mathbb{R}^d, \mathbb{R})$ by $B_\infty[0, M]$. The following lemma will illuminate why we have engaged in defining these objects.

Lemma 20. *Assume A.1, A.4, A.5, A.6 and A.7.*

1. For every (r, τ) , such that $r \in [0, T[, \tau \in]0, T - r]$, $L^1([r, r + \tau], B[0, M]) \cap B_\infty[0, M]$ is a closed subset of $L^1([r, r + \tau] \times \mathbb{R}^d)$. In particular, it is a complete metric space equipped with $\|\cdot\|_1$, defined in (28), with $t_1 = r$ and $t_2 = r + \tau$.

2. There exists $\tau > 0$ such that for any $r \in [0, T - \tau]$, we have

$$\Pi(L^1([r, r + \tau], B[0, M]) \cap B_\infty[0, M]) \subset (L^1([r, r + \tau], B[0, M]) \cap B_\infty[0, M]).$$

3. Let τ be as in previous item. There exists an integer $k_0 \in \mathbb{N}$, such that Π^{k_0} is a contraction on $L^1([r, r + \tau], B[0, M]) \cap B_\infty[0, M]$ for any $r \in [0, T - \tau]$.

In particular, for any $r \in [0, T - \tau]$, there exists a unique fixed-point of Π . This fixed-point is in $L^1([r, r + \tau], B[0, M]) \cap B_\infty[0, M]$.

Proof. 1. As for the first claim, we note that $L^1([r, r + \tau], B[0, M])$ is a complete space with respect to the norm defined in (28). Hence it suffices to show that $B_\infty[0, M]$ is closed with respect to this norm. So let $(f_n)_{n \in \mathbb{N}} \subset B_\infty[0, M]$ be such that $f_n \rightarrow f$ with respect to (28). Then we can extract a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ which converges almost everywhere pointwise on $[0, T] \times \mathbb{R}^d$, against f . Because (f_n) are uniformly bounded by M , so is f . Now we only need to recall that the intersection of closed sets (in this case $L^1([r, r + \tau], B[0, M])$ and $B_\infty[0, M]$) is closed.

2. Let (r, τ) , be fixed for the moment. Without loss of generality we assume $M = 1$. Let $v \in L^1([r, r + \tau], B[0, 1]) \cap B_\infty[0, 1]$. Now, the triangle inequality yields

$$\begin{aligned} \|\Pi(v)\|_1 &= \int_r^{r+\tau} \|\Pi(v)(t, \cdot)\|_{L^1(\mathbb{R}^d)} dt \\ &\leq \int_r^{r+\tau} \left\| \int_r^t \int_{\mathbb{R}^d} p(s, x_0, t, \cdot) \hat{\Lambda}(s, x_0, v + \hat{u}_0(\phi)) dx_0 ds \right\|_{L^1(\mathbb{R}^d)} dt \\ &\quad + \sum_{j=1}^d \int_r^{r+\tau} \left\| \int_r^t \int_{\mathbb{R}^d} \partial_{x_{0,j}} p(s, x_0, t, \cdot) \hat{b}_j(s, x_0, v + \hat{u}_0(\phi)) dx_0 ds \right\|_{L^1(\mathbb{R}^d)} dt. \end{aligned}$$

Note that for the first term on the right-hand side of this inequality, we already have an upper bound from equation (3.28) in the proof of Lemma 3.7 in [22] which reads

$$\int_r^{r+\tau} \left\| \int_r^t \int_{\mathbb{R}^d} p(s, x_0, t, \cdot) \hat{\Lambda}(s, x_0, v + \hat{u}_0(\phi)) dx_0 ds \right\|_{L^1(\mathbb{R}^d)} dt \leq 2M_\Lambda \tau^2. \quad (32)$$

As for the second term, we restrict our discussion to the first summand, $j = 1$, because all summands can be estimated in the same way. We adapt the estimation of the $W^{1,1}$ -norm in the proof of [22, Lemma 3.7] :

$$\int_{\mathbb{R}^d} \left| \int_r^t \int_{\mathbb{R}^d} \partial_{x_{0,1}} p(s, x_0, t, x) \hat{b}_1(s, x_0, v + \hat{u}_0(\phi)) dx_0 ds \right| dx$$

$$\begin{aligned}
&\leq M_b \int_{\mathbb{R}^d} \int_r^t \int_{\mathbb{R}^d} |\partial_{x_0,1} p(s, x_0, t, x)| |(v + \widehat{u}_0(\phi))(s, x_0)| dx_0 ds dx \\
&\stackrel{\text{Tonelli}}{=} M_b \int_r^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_{x_0,1} p(s, x_0, t, x)| |(v + \widehat{u}_0(\phi))(s, x_0)| dx dx_0 ds \\
&\stackrel{\text{ineq.(12)}}{\leq} M_b C_u \int_r^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^d} |(v + \widehat{u}_0(\phi))(s, x_0)| \int_{\mathbb{R}^d} q(s, x_0, t, x) dx dx_0 ds \\
&= M_b C_u \int_r^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^d} |(v + \widehat{u}_0(\phi))(s, x_0)| dx_0 ds \\
&= M_b C_u \int_r^t \frac{1}{\sqrt{t-s}} \underbrace{\|(v + \widehat{u}_0(\phi))(s, \cdot)\|_{L^1(\mathbb{R}^d)}}_{\leq \|v(s, \cdot)\|_{L^1(\mathbb{R}^d)} + \|\widehat{u}_0(\phi)(s, \cdot)\|_{L^1(\mathbb{R}^d)} \leq 1 + \|\phi\|_1} ds \\
&\leq (1 + \|\phi\|_1) M_b C_u \int_r^t \frac{1}{\sqrt{t-s}} ds \leq 2(1 + \|\phi\|_1) M_b C_u \sqrt{\tau}.
\end{aligned}$$

This and (32) give

$$\|\Pi(v)\|_1 \leq 2M_\Lambda \tau^2 + 4dM_b C_u \sqrt{\tau} = 2\sqrt{\tau}(M_\Lambda \tau^{\frac{3}{2}} + 2dM_b C_u). \quad (33)$$

For the L^∞ -norm of $\Pi(v)$, we repeat the same calculation, which yields that there also is a constant $\bar{C} := \bar{C}(C_u, c_u, \Lambda, b, d) > 0$ such that

$$\|\Pi(v)\|_\infty \leq \bar{C} \sqrt{\tau}. \quad (34)$$

In particular, $\|\Pi(v)\|_1$ and $\|\Pi(v)\|_\infty$ converge to zero when τ converges to zero. So we can choose τ small enough to ensure that $\|\Pi(v)\|_1 \leq 1$ and $\|\Pi(v)\|_\infty \leq 1$. This completes the proof of claim 2.

3. To verify the contraction property on $L^1([r, r + \tau], B[0, M]) \cap B_\infty[0, M]$ for any $r \in [0, T - \tau]$, we fix $v_1, v_2 \in L^1([r, r + \tau], B[0, M]) \cap B_\infty[0, M]$ and $t \in [0, T - \tau]$. For readability, we use the shorthand notation $\Lambda^i(s, x_0) := \Lambda(s, x_0, (\widehat{u}_0(r, \phi) + v_i)(s, x_0))$ and similarly $b_j^i(s, x_0) := \hat{b}_j(s, x_0, (\widehat{u}_0(r, \phi) + v_i)(s, x_0))$ for $i = 1, 2$ and $1 \leq j \leq d$. Then, for $t \in [r, r + \tau]$, we compute

$$\begin{aligned}
&\|\Pi(v_1)(t, \cdot) - \Pi(v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \\
&\leq \int_{\mathbb{R}^d} \left| \int_r^t \int_{\mathbb{R}^d} p(s, x_0, t, x) (\Lambda^1(s, x_0)(\widehat{u}_0 + v_1)(s, x_0) - \Lambda^2(s, x_0)(\widehat{u}_0 + v_2)(s, x_0)) dx_0 ds \right| dx \\
&\quad + \sum_{j=1}^d \int_{\mathbb{R}^d} \left| \int_r^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) (b_j^1(s, x_0)(\widehat{u}_0 + v_1)(s, x_0) - b_j^2(s, x_0)(\widehat{u}_0 + v_2)(s, x_0)) dx_0 ds \right| dx
\end{aligned}$$

$$=: I_1 + I_2.$$

For the rest of the proof, we will forget the argument (s, x_0) for simplicity. Concerning I_1 , we compute as follows:

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} \left| \int_r^t \int_{\mathbb{R}^d} p(s, x_0, t, x) (\Lambda^1(\widehat{u}_0 + v_1) - \Lambda^2(\widehat{u}_0 + v_2)) dx_0 ds \right| dx \\
&\leq \int_{\mathbb{R}^d} \int_r^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \left| (\Lambda^1(\widehat{u}_0 + v_1) - \Lambda^2(\widehat{u}_0 + v_2)) \right| dx_0 ds dx \\
&= \int_r^t \int_{\mathbb{R}^d} \left| (\Lambda^1(\widehat{u}_0 + v_1) - \Lambda^2(\widehat{u}_0 + v_2)) + \Lambda^1 v_2 - \Lambda^1 v_2 \right| dx_0 ds \\
&= \int_r^t \int_{\mathbb{R}^d} \left| (\Lambda^1 - \Lambda^2)\widehat{u}_0 + \Lambda^1(v_1 - v_2) + (\Lambda^1 - \Lambda^2)v_2 \right| dx_0 ds \\
&\leq \int_r^t \int_{\mathbb{R}^d} \underbrace{|\Lambda^1 - \Lambda^2|}_{\leq L_\Lambda |v_1 - v_2|} |\widehat{u}_0| + \underbrace{|\Lambda^1|}_{\leq M_\Lambda} |v_1 - v_2| + \underbrace{|\Lambda^1 - \Lambda^2|}_{\leq L_\Lambda |v_1 - v_2|} \underbrace{|v_2|}_{\leq M} dx_0 ds \\
&= (2L_\Lambda M + M_\Lambda) \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds.
\end{aligned}$$

For I_2 , we again restrict our discussion to the first summand. We can estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \int_r^t \int_{\mathbb{R}^d} \partial_{x_0,1} p(s, x_0, t, x) (b_1^1(\widehat{u}_0 + v_1) - b_1^2(\widehat{u}_0 + v_2)) dx_0 ds \right| dx \\
&\leq \int_r^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b_1^1(\widehat{u}_0 + v_1) - b_1^2(\widehat{u}_0 + v_2)| |\partial_{x_0,1} p(s, x_0, t, x)| dx dx_0 ds \\
&\stackrel{(12)}{\leq} \int_r^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^d} |b_1^1(\widehat{u}_0 + v_1) - b_1^2(\widehat{u}_0 + v_2)| \int_{\mathbb{R}^d} C_u q(s, x_0, t, x) dx dx_0 ds \\
&\leq C_u \int_r^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^d} |b_1^1(\widehat{u}_0 + v_1) - b_1^2(\widehat{u}_0 + v_2)| dx_0 ds.
\end{aligned}$$

By similar estimates as for I_1 , previous expression can be shown to be bounded by

$$C_u (2L_b M + M_b) \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds.$$

Until now, we have established that there exists a constant $C > 0$ which can change from line to line such that

$$\begin{aligned} \|\Pi(v_1)(t, \cdot) - \Pi(v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)} &\leq C \int_r^t \left(1 + \frac{1}{\sqrt{t-s}}\right) \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds \\ &\leq C \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds. \end{aligned} \quad (35)$$

We now try to find a power of Π which is a contraction. For this, we iterate Π , aiming to apply a J.B. Walsh iteration trick, see e.g. the proof of Theorem 3.2 in [31]. For $t \in [r, r + \tau]$ we have

$$\begin{aligned} &\|\Pi^2(v_1)(t, \cdot) - \Pi^2(v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \\ &\leq C^2 \int_r^t \int_r^s \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{L^1(\mathbb{R}^d)} d\theta ds \\ &= C^2 \int_r^t \int_\theta^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{L^1(\mathbb{R}^d)} ds d\theta \\ &= C^2 \int_r^t \left(\int_\theta^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} ds \right) \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{L^1(\mathbb{R}^d)} d\theta \\ &= C^2 \int_r^t \left(\int_0^{t-\theta} \frac{1}{\sqrt{(t-\theta)-\omega}} \frac{1}{\sqrt{\omega}} d\omega \right) \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{L^1(\mathbb{R}^d)} d\theta \\ &= \pi C^2 \int_r^t \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{L^1(\mathbb{R}^d)} d\theta. \end{aligned} \quad (36)$$

Indeed the last equality follows because

$$\int_0^{t-\theta} \frac{1}{\sqrt{(t-\theta)-\omega}} \frac{1}{\sqrt{\omega}} d\omega = B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi,$$

where $(x, y) \mapsto B(x, y)$ is the Beta function.

Actually, this is the induction basis of a mathematical induction over $k \in \mathbb{N}$ to establish

$$\|\Pi^{2k}(v_1)(t, \cdot) - \Pi^{2k}(v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \pi^k C^{2k} \int_r^t \frac{(t-s)^{k-1}}{(k-1)!} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds. \quad (37)$$

So let us do the inductive step. The third line comes from (36).

$$\begin{aligned}
& \|\Pi^{2k+2}(v_1)(t, \cdot) - \Pi^{2k+2}(v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \\
&= \|\Pi^2 \Pi^{2k}(v_1)(t, \cdot) - \Pi^2 \Pi^{2k}(v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \\
&\leq \pi C^2 \int_r^t \|\Pi^{2k}(v_1)(\theta, \cdot) - \Pi^{2k}(v_2)(\theta, \cdot)\|_{L^1(\mathbb{R}^d)} d\theta \\
&\leq \pi C^2 \int_r^t \pi^k C^{2k} \int_r^\theta \frac{(\theta - s)^{k-1}}{(k-1)!} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds d\theta \\
&\leq \pi^{k+1} C^{2(k+1)} \int_r^t \int_s^t \frac{(\theta - s)^{k-1}}{(k-1)!} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} d\theta ds \\
&= \pi^{k+1} C^{2(k+1)} \int_r^t \frac{1}{(k-1)!} \int_s^t (\theta - s)^{k-1} d\theta \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds \\
&= \frac{\pi^{k+1} C^{2(k+1)}}{(k-1)!} \int_r^t \frac{(t-s)^k}{k} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds.
\end{aligned}$$

Now that we have established equation (37), we can use it to show that a power of Π is a contraction:

$$\|\Pi^{2k}(v_1)(t, \cdot) - \Pi^{2k}(v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \frac{\pi^k T^{k-1} C^{2k}}{(k-1)!} \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{L^1(\mathbb{R}^d)} ds.$$

So, integrating from r to $r + \tau$, we get

$$\|\Pi^{2k}(v_1) - \Pi^{2k}(v_2)\|_1 \leq \frac{\pi^k T^k C^{2k}}{(k-1)!} \|v_1 - v_2\|_1. \quad (38)$$

Now we can conclude that there exists a $k_0 \in \mathbb{N}$ such that Π^{2k_0} is a contraction because the exponential growth of $\pi^k T^k C^{2k}$ is dominated by the factorial $(k-1)!$ Finally, Π admits a unique fixed point as a contraction on a Banach space by Banach's fixed point theorem. \square

Now that we have constructed fixed points for short time intervals, we are concerned with “gluing” these fixed points together.

Lemma 21. *We fix some bounded Borel measurable functions $\hat{\Lambda} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\hat{b} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. We suppose A.4. Fix $\tau > 0$ such that there is an $N \in \mathbb{N}$ s.t. $N\tau = T$. Then*

a measure valued map $\mu : [0, T] \mapsto \mathcal{M}_f(\mathbb{R}^d)$ satisfies

$$\left\{ \begin{array}{l} \mu_t(dx) = \int_{\mathbb{R}^d} p(k\tau, x_0, t, x) \mu_{k\tau}(dx_0) dx \\ \quad + \int_{k\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\ \quad + \sum_{j=1}^d \int_{k\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx, \\ \mu_0(\cdot) = \mathbf{u}_0, \end{array} \right. \quad (39)$$

for all $t \in [k\tau, (k+1)\tau]$ and $k \in \{0, \dots, N-1\}$ if and only if μ is a measure-mild solution of (26), in the sense of Definition 18.

Proof. 1. Assume first that μ satisfies (39) for all $t \in [k\tau, (k+1)\tau]$ and $k \in \{0, \dots, N-1\}$.

We will now show that this implies that μ is a measure-mild solution of (26) in the sense of (27). We will perform a mathematical induction to show that the statement

$$(S_n) \left\{ \begin{array}{l} \mu_t(dx) = \int_{\mathbb{R}^d} p(0, x_0, t, x) \mathbf{u}_0(dx_0) dx + \int_0^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\ \quad + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx \quad \text{for all } t \in [0, n\tau] \end{array} \right.$$

holds for any $n \in \{1, \dots, N\}$. For the induction basis, it suffices to note that if we set $k = 0$, equation (27) coincides with (39). Now suppose that (S_{n-1}) holds for some $n \in \{1, \dots, N\}$. In particular, we have for $t = (n-1)\tau$

$$\begin{aligned} \mu_{(n-1)\tau}(dx_0) &= \int_{\mathbb{R}^d} p(0, \tilde{x}_0, (n-1)\tau, x_0) \mathbf{u}_0(d\tilde{x}_0) dx_0 \\ &\quad + \int_0^{(n-1)\tau} \int_{\mathbb{R}^d} p(s, \tilde{x}_0, (n-1)\tau, x_0) \hat{\Lambda}(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0 \\ &\quad + \sum_{j=1}^d \int_0^{(n-1)\tau} \int_{\mathbb{R}^d} \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, (n-1)\tau, x_0) \hat{b}_j(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0. \end{aligned} \quad (40)$$

Recall that we also suppose the validity of equation (39), i.e. for $k = n-1$ and $t \in [(n-1)\tau, n\tau]$ we have

$$\begin{aligned} \mu_t(dx) &= \int_{\mathbb{R}^d} p((n-1)\tau, x_0, t, x) \mu_{(n-1)\tau}(dx_0) dx \\ &\quad + \int_{(n-1)\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \end{aligned}$$

$$+ \sum_{j=1}^d \int_{(n-1)\tau}^t \int_{\mathbb{R}^d} \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, t, x) \hat{b}_j(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx. \quad (41)$$

Now, plugging equation (40) into equation (41), we find

$$\begin{aligned} & \mu_t(dx) \quad (42) \\ &= \int_{\mathbb{R}^d} p((n-1)\tau, x_0, t, x) \left(\int_{\mathbb{R}^d} p(0, \tilde{x}_0, (n-1)\tau, x_0) \mathbf{u}_0(d\tilde{x}_0) dx_0 \right. \\ & \quad + \int_0^{(n-1)\tau} \int_{\mathbb{R}^d} p(s, \tilde{x}_0, (n-1)\tau, x_0) \hat{\Lambda}(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0 \\ & \quad + \left. \sum_{j=1}^d \int_0^{(n-1)\tau} \int_{\mathbb{R}^d} \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, (n-1)\tau, x) \hat{b}_j(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0 \right) dx \\ & \quad + \int_{(n-1)\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\ & \quad + \sum_{j=1}^d \int_{(n-1)\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p(0, \tilde{x}_0, (n-1)\tau, x_0) p((n-1)\tau, x_0, t, x) dx_0 \right) \mathbf{u}_0(d\tilde{x}_0) dx \\ & \quad + \left[\int_0^{(n-1)\tau} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p(s, \tilde{x}_0, (n-1)\tau, x_0) p((n-1)\tau, x_0, t, x) dx_0 \right) \hat{\Lambda}(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx \right. \\ & \quad + \left. \int_{(n-1)\tau}^t \int_{\mathbb{R}^d} p(s, \tilde{x}_0, t, x) \hat{\Lambda}(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx \right] \\ & \quad + \left[\int_0^{(n-1)\tau} \sum_{j=1}^d \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, (n-1)\tau, x_0) p((n-1)\tau, x_0, t, x) dx_0 \right) \hat{b}_j(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx \right. \\ & \quad + \left. \sum_{j=1}^d \int_{(n-1)\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, \tilde{x}_0, t, x) \hat{b}_j(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx \right]. \quad (43) \end{aligned}$$

Now, focus on the right-hand side of the equation. For the first two terms, we can now use the Chapman-Kolmogorov identity (see (13)) for p which is included in Assumption A.4. That is, for a.e. (t, x) and (s, x_0) , we have

$$\int_{\mathbb{R}^d} p(s, \tilde{x}_0, (n-1)\tau, x_0) p((n-1)\tau, x_0, t, x) dx_0 = p(s, \tilde{x}_0, t, x). \quad (44)$$

We differentiate (44) with respect to x_0 , we integrate the resulting identity against a test function and use Tonelli's theorem to conclude that

$$\int_{\mathbb{R}^d} \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, (n-1)\tau, x_0) p((n-1)\tau, x_0, t, x) dx_0 = \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, t, x), \quad (t, x), (s, x_0), \quad \text{a.e.} \quad (45)$$

for any $j \in \{1, \dots, d\}$. Applying (44) and (45) to the right-hand side of (43) finally yields that μ is a measure-mild solution of (26) in the sense of Definition 18. i.e. establishes S_n .

2. Now suppose that μ is a measure-mild solution of (26) in the sense of Definition 18.

We use a mathematical induction over $k \in \{0, 1, \dots, N-1\}$ to show that (39) holds. Again, the induction basis is obvious because for $k = 0$, (39) and (27) coincide. Now suppose that (39) holds for all $k \in \{0, 1, \dots, n-1\}$. Recall the induction hypothesis for $k = (n-1)$ and $t = n\tau$:

$$\begin{aligned} \mu_{n\tau}(dx) &= \int_{\mathbb{R}^d} p((n-1)\tau, x_0, n\tau, x) \mu_{(n-1)\tau}(dx_0) dx \\ &\quad + \int_{(n-1)\tau}^{n\tau} \int_{\mathbb{R}^d} p(s, x_0, n\tau, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\ &\quad + \sum_{j=1}^d \int_{(n-1)\tau}^{n\tau} \int_{\mathbb{R}^d} \partial_{x_0} p(s, x_0, n\tau, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx. \end{aligned}$$

We begin by exploiting the definition of a measure-mild solution of (26) in the sense of Definition 18:

$$\begin{aligned} &\int_{\mathbb{R}^d} p(n\tau, x_0, t, x) \mu_{n\tau}(dx_0) dx + \int_{n\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\ &\quad + \sum_{j=1}^d \int_{n\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx \\ &= \int_{\mathbb{R}^d} p(n\tau, x_0, t, x) \left(\int_{\mathbb{R}^d} p(0, \tilde{x}_0, n\tau, x_0) \mu_0(d\tilde{x}_0) dx_0 \right. \\ &\quad + \int_0^{n\tau} \int_{\mathbb{R}^d} p(s, \tilde{x}_0, n\tau, x_0) \hat{\Lambda}(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0 \\ &\quad \left. + \sum_{j=1}^d \int_0^{n\tau} \int_{\mathbb{R}^d} \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, n\tau, x_0) \hat{b}_j(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0 \right) \end{aligned}$$

$$\begin{aligned}
& + \int_{n\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\
& + \sum_{j=1}^d \int_{n\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx \\
= & \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p(0, \tilde{x}_0, n\tau, x_0) p(n\tau, x_0, t, x) dx_0 \right) \mu_0(d\tilde{x}_0) \\
& + \int_0^{n\tau} \int_{\mathbb{R}^d} p(s, \tilde{x}_0, n\tau, x_0) \hat{\Lambda}(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0 \\
& + \sum_{j=1}^d \int_0^{n\tau} \int_{\mathbb{R}^d} \partial_{\tilde{x}_0, j} p(s, \tilde{x}_0, n\tau, x_0) \hat{b}_j(s, \tilde{x}_0) \mu_s(d\tilde{x}_0) ds dx_0 \\
& + \int_{n\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\
& + \sum_{j=1}^d \int_{n\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx \\
= & \int_{\mathbb{R}^d} p(0, \tilde{x}_0, t, x) \mu_0(d\tilde{x}_0) + \int_0^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \hat{\Lambda}(s, x_0) \mu_s(dx_0) ds dx \\
& + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) \hat{b}_j(s, x_0) \mu_s(dx_0) ds dx \stackrel{(27)}{=} \mu_t(dx).
\end{aligned}$$

Thus the claim of the inductive step is established and the proof accomplished. \square

With these results, we can now formulate the central existence and uniqueness theorem. The idea will be to find local "mild solutions" by Lemma 20 and then to "glue them together" by means of Lemma 21.

Theorem 22. *Let Assumptions A.1, A.4, A.6 and A.7 be in force.*

1. *Under Assumption A.5, there exists a unique bounded mild solution of (2).*
2. *Under Assumption C5, there is at most one bounded mild solution of (2).*

Proof. (a) Existence. We assume Assumption A.5. To begin constructing the local "mild solutions", we set $r = 0$, $\tau > 0$ such that $T = N\tau$. We also set $\phi = u_0$ in (29), where $\mathbf{u}_0(dx_0) = u_0(x_0) dx_0$. Recall that we have assumed u_0 to be a bounded density, so both

its L^∞ -norm and its L^1 -norm can be bounded by a constant $M \geq 1$. First, we define for $(t, x) \in [0, \tau] \times \mathbb{R}^d$

$$\widehat{u}_0^0(t, x) := \widehat{u}_0(0, u_0)(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0. \quad (46)$$

Now, we can apply Lemma 20 to infer the existence of a fixed-point of the mapping $\Pi : L^1([0, \tau], L^1(\mathbb{R}^d)) \rightarrow L^1([0, \tau], L^1(\mathbb{R}^d))$, defined by

$$\begin{aligned} \Pi(r, v)(t, x) := & \int_r^t \int_{\mathbb{R}^d} p(s, x_0, t, x) (v + \widehat{u}_0(\phi))(s, x_0) \Lambda(s, x_0, (v + \widehat{u}_0)(\phi)(s, x_0)) dx_0 ds \\ & + \sum_{j=1}^d \int_r^t \int_{\mathbb{R}^d} \partial_{x_{0,j}} p(s, x_0, t, x) (v + \widehat{u}_0(\phi))(s, x_0) b_j(s, x_0, (v + \widehat{u}_0(\phi)(s, x_0))) dx_0 ds. \end{aligned} \quad (47)$$

We will refer to this fixed-point as v^0 . Note that $v^0 \in L^1([0, \tau], B(0, M)) \cap B_\infty(0, M)$. Then, we set $u^0(t, x) = \widehat{u}_0(t, x) + v^0(t, x)$ for $(t, x) \in [0, \tau] \times \mathbb{R}^d$. Note that u^0 satisfies equation (15) from the definition of mild solution restricted on $t \in [0, \tau]$.

To extend the construction of u^0 for values of t above τ , we proceed by induction. Fix some $k \in \{1, \dots, N-1\}$ and suppose we are given a family of functions u^1, u^2, \dots, u^{k-1} such that for any $\ell \in \{1, \dots, k-1\}$ it holds that $u^\ell \in L^1([\ell\tau, (\ell+1)\tau], L^1(\mathbb{R}^d)) \cap L^\infty([\ell\tau, (\ell+1)\tau] \times \mathbb{R}^d, \mathbb{R})$ and u^ℓ satisfies

$$\begin{aligned} u^\ell(t, x) = & \int_{\mathbb{R}^d} p(\ell\tau, x_0, t, x) u^{\ell-1}(\ell\tau, x_0) dx \\ & + \int_{\ell\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \Lambda(s, x_0, u^\ell(s, x_0)) u^\ell(s, x_0) dx_0 ds dx \\ & + \sum_{j=1}^d \int_{\ell\tau}^t \int_{\mathbb{R}^d} \partial_{x_{0,j}} p(s, x_0, t, x) b_j(s, x_0, u^\ell(s, x_0)) u^\ell(s, x_0) dx_0 ds dx, \end{aligned} \quad (48)$$

for all $(t, x) \in [\ell\tau, (\ell+1)\tau] \times \mathbb{R}^d$. In order to define u^k , we begin by defining \widehat{u}_0^k

$$\widehat{u}_0^k(t, x) := \widehat{u}_0(k\tau, u^{k-1}(k\tau, \cdot))(t, x) = \int_{\mathbb{R}^d} p(k\tau, x_0, t, x) u^{k-1}(k\tau, x_0) dx_0,$$

for $(t, x) \in [k\tau, (k+1)\tau]$. Again, we can choose M large enough so as to satisfy $M \geq \max\{\|u_{k-1}(k\tau, \cdot)\|_\infty, \|u_{k-1}(k\tau, \cdot)\|_1\}$. Then set $r = k\tau$ and $\phi = u^{k-1}(k\tau, \cdot)$ which allows us to apply Lemma 20 thereby establishing existence and uniqueness of a function v^k :

$[k\tau, (k+1)\tau] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ that belongs to $L^1([k\tau, (k+1)\tau], B(0, M)) \cap B_\infty(0, M)$ and satisfying

$$\begin{aligned} v^k(t, x) &= \int_{k\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \Lambda(s, x_0, u^k(s, x_0)) u^k(s, x_0) dx_0 ds dx \\ &\quad + \sum_{j=1}^d \int_{k\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) b_j(s, x_0, u^k(s, x_0)) u^k(s, x_0) dx_0 ds dx, \end{aligned}$$

for all $(t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d$. Again, we define $u^k(t, x) := \widehat{u}_0^k(t, x) + v^k(t, x)$ for $(t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d$. In particular, for $(t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d$ we have

$$\begin{aligned} u^k(t, x) &= \int_{\mathbb{R}^d} p(k\tau, x_0, t, x) u^{k-1}(k\tau, x_0) dx \\ &\quad + \int_{k\tau}^t \int_{\mathbb{R}^d} p(s, x_0, t, x) \Lambda(s, x_0, u^k(s, x_0)) u^k(s, x_0) dx_0 ds dx \\ &\quad + \sum_{j=1}^d \int_{k\tau}^t \int_{\mathbb{R}^d} \partial_{x_0, j} p(s, x_0, t, x) b_j(s, x_0, u^k(s, x_0)) u^k(s, x_0) dx_0 ds dx. \end{aligned}$$

This shows that we can indeed use induction to define a sequence u^1, u^2, \dots, u^N of functions such that for any $\ell \in \{1, \dots, N-1\}$ we have $u^\ell \in L^1([\ell\tau, (\ell+1)\tau], L^1(\mathbb{R}^d)) \cap L^\infty([\ell\tau, (\ell+1)\tau] \times \mathbb{R}^d, \mathbb{R})$ which satisfies (48) for any $(t, x) \in [\ell\tau, (\ell+1)\tau] \times \mathbb{R}^d$.

Thus we can define a function u on $[0, T] \times \mathbb{R}^d$ by setting $u(t, x) = u^\ell(t, x)$ if $t \in [\ell\tau, (\ell+1)\tau]$. Invoking Lemma 21 with

$$\widehat{\Lambda}(s, x) = \Lambda(s, x, u(s, x)), \quad \widehat{b}(s, x) = b(s, x, u(s, x)), \quad \mu_t(dx) := u(t, x) dx,$$

shows that μ is a measure-mild solution of (26). Recalling Remark 19 we see that u is a mild solution of (2).

- (b) Uniqueness. Here we only suppose the validity of Assumption C5. To establish uniqueness of a bounded mild solution, suppose that u_1 and u_2 are two bounded mild solutions of (2) in the sense of Definition 6. Then with similar estimations as for proving (3), we find that there exists a constant C , depending only on $u_1, u_2, \Lambda, \Phi, b$ and b_0 such that for all $\ell \in \{0, \dots, N-1\}$, we have

$$\int_0^T \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1} dt \leq \frac{C^\ell}{\ell!} \int_0^T \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1} dt,$$

from which we can conclude that $\int_0^T \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1} dt = 0$.

□

6 Uniqueness for the linear PDE

We fix the same framework as the one of Section 5.1.

6.1 Uniqueness of measure-mild solutions

Proposition 23. *Assume A.4 and that $\hat{b} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\hat{\Lambda} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ are measurable and bounded. Then there is at most one measure-mild solution of (26).*

Proof. Suppose that μ_1 and μ_2 are measure-mild solutions of (26). Then we define $\nu := \mu_1 - \mu_2$. Note that the definition of measure-mild solutions implies that ν satisfies

$$\begin{aligned} \nu_t(dx) &= \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) p(r, x_0, t, x) dx \nu_r(dx_0) dr \\ &\quad + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \hat{b}_j(r, x_0) \partial_{x_{0,j}} p(r, x_0, t, x) dx \nu_r(dx_0) dr. \end{aligned}$$

We prepare the application of Gronwall's Lemma estimating the total variation norm of $\nu_t(\cdot)$.

Step 1. Estimation of the total variation norm of ν

Now fix an arbitrary $t \in [0, T]$. Then we have

$$\begin{aligned} \|\nu_t(\cdot)\|_{TV} &= \sup_{\substack{\Psi \in C_b(\mathbb{R}^d) \\ \|\Psi\|_\infty \leq 1}} \left| \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) \left(\int_{\mathbb{R}^d} \Psi(x) p(r, x_0, t, x) dx \right) \nu_r(dx_0) dr \right. \\ &\quad \left. + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \hat{b}_j(r, x_0) \left(\int_{\mathbb{R}^d} \Psi(x) \partial_{x_{0,j}} p(r, x_0, t, x) dx \right) \nu_r(dx_0) dr \right| \\ &\leq \sup_{\substack{\Psi \in C_b(\mathbb{R}^d) \\ \|\Psi\|_\infty \leq 1}} \left| \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) \left(\int_{\mathbb{R}^d} \Psi(x) p(r, x_0, t, x) dx \right) \nu_r(dx_0) dr \right| \\ &\quad + \sup_{\substack{\Psi \in C_b(\mathbb{R}^d) \\ \|\Psi\|_\infty \leq 1}} \sum_{j=1}^d \left| \int_0^t \int_{\mathbb{R}^d} \hat{b}_j(r, x_0) \left(\int_{\mathbb{R}^d} \Psi(x) \partial_{x_{0,j}} p(r, x_0, t, x) dx \right) \nu_r(dx_0) dr \right| \\ &\leq \sup_{\substack{\Psi \in C_b(\mathbb{R}^d) \\ \|\Psi\|_\infty \leq 1}} \int_0^t \int_{\mathbb{R}^d} |\hat{\Lambda}(r, x_0)| \left(\int_{\mathbb{R}^d} \underbrace{|\Psi(x)|}_{\leq 1} p(r, x_0, t, x) dx \right) |\nu_r|(dx_0) dr \\ &\quad + \sup_{\substack{\Psi \in C_b(\mathbb{R}^d) \\ \|\Psi\|_\infty \leq 1}} \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} |\hat{b}_j(r, x_0)| \left(\int_{\mathbb{R}^d} \underbrace{|\Psi(x)|}_{\leq 1} \underbrace{|\partial_{x_{0,j}} p(r, x_0, t, x)|}_{\leq C_u \frac{1}{\sqrt{t-r}} q(r, x_0, t, x)}}_{(12)} dx \right) |\nu_r|(dx_0) dr \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^d} \int_0^t \underbrace{|\hat{\Lambda}(r, x_0)|}_{\leq M_\Lambda} \underbrace{\int_{\mathbb{R}^d} p(r, x_0, t, x) dx}_{=1} |\nu_r|(dx_0) dr \\
&\quad + C_u \int_{\mathbb{R}^d} \int_0^t \underbrace{|\hat{b}_j(r, x_0)|}_{\leq M_b} \frac{1}{\sqrt{t-r}} \underbrace{\int_{\mathbb{R}^d} q(r, x_0, t, x) dx}_{=1} |\nu_r|(dx_0) dr \\
&\leq C_1 \int_0^t \left(1 + \frac{1}{\sqrt{t-r}}\right) \int_{\mathbb{R}^d} |\nu_r|(dx_0) dr \\
&= C_1 \int_0^t \left(1 + \frac{1}{\sqrt{t-r}}\right) \|\nu_r\|_{TV} dr,
\end{aligned}$$

where we have used inequality (12), the fact that $x \mapsto p(s, x_0, t, x)$ is a probability density and the definition of the total variation norm. For future reference, we note that we have shown

$$\begin{aligned}
\|\nu_t\|_{TV} &\leq C_1 \int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right) \|\nu_s\|_{TV} ds, \\
&\leq C \int_0^t \frac{1}{\sqrt{t-s}} \|\nu_s\|_{TV} ds,
\end{aligned}$$

for $C := C_1(T+1)$.

Step 2. Iterating the estimation and Gronwall lemma

Similarly to the estimates from (35) to (36), (49) allows to prove

$$\|\nu_t\|_{TV} \leq \pi C^2 \int_0^t \|\nu_r\|_{TV} dr. \tag{49}$$

Applying Gronwall's Lemma on the basis of (49), we infer that $\|\nu_t\|_{TV} = 0$. Because this holds for any $t \in [0, T]$ and since $\nu = \mu_1 - \mu_2$, this is exactly the claimed uniqueness of measure-mild solutions. \square

6.2 Equivalence weak-mild for the linear PDE

Similarly as for the nonlinear PDE where we used the notion of weak solution, see Definition 5, we shall also need to introduce the concept of measure-weak solutions of (26).

Definition 24. Assume [A.1](#) and that \hat{b} and $\hat{\Lambda}$ are bounded. We then call $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ a *measure-weak* solution of [\(26\)](#) if we have for all $\varphi \in C_0^\infty$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) &= \int_{\mathbb{R}^d} \varphi(x) \mathbf{u}_0(dx) + \sum_{j=1}^d \int_{\mathbb{R}^d} \int_0^t \partial_j \varphi(x) \hat{b}(s, x) \mu_s(dx) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} L_s \varphi(x) \mu_s(dx) ds + \int_0^t \int_{\mathbb{R}^d} \varphi(x) \hat{\Lambda}(s, x) \mu_s(dx) ds. \end{aligned}$$

We can naturally adapt [Proposition 16](#) to the linearized case.

Proposition 25. We assume [A.1](#), [A.4](#), [C](#) and that $\hat{b}, \hat{\Lambda}$ are bounded. Let $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$. μ is a *measure-mild* solution of [\(26\)](#) if and only if μ is a *measure-weak* solution.

Proof. The proof is analogous to the proof of [Proposition 16](#) if one keeps the second statement of [Remark 19](#) in mind. \square

7 Appendix

7.1 A sufficient condition for Assumption [A.4](#)

Proposition 26. Let us suppose the following properties.

1. For all $i, j \in \{1, \dots, d\}$ for all $t \in [0, T]$, the mapping $a_{ij}(t, \cdot)$ is in C^2 and the mapping $b_0(t, \cdot)$ is in C^1 whose derivatives with respect to the space variable x are bounded on $[0, T] \times \mathbb{R}^d$.
2. The partial derivatives of order 2 (resp. order 1) for the components of a (resp. b_0) are Hölder continuous in space, uniformly with respect to time, with some parameter α .
3. Assumptions [A.1](#) and [A.2](#) hold.

Then Assumption [A.4](#) is verified. In particular there is a Markov fundamental solution of [\(10\)](#), in the sense of [Definition 2](#).

Proof. In the definition stated in sect. 1, p.3 of [\[13\]](#)) Friedman considers a notion of fundamental solution of

$$\partial_t u = L_t^* u, \tag{50}$$

$$\partial_t u = L_t u. \tag{51}$$

By Theorems 15, section 9, chap. 1 in [\[13\]](#) and inequalities [\(8.13\)](#) and [\(8.14\)](#) just before, there exist fundamental solutions p, Γ in the sense of Friedman of [\(50\)](#), [\(51\)](#) fulfilling [\(11\)](#) and [\(12\)](#) such that

$$p(s, x_0, t, x) = \Gamma(T - t, x, t - s, x_0). \tag{52}$$

We can verify then that p is a fundamental solution of [\(7\)](#) in the sense of [Definition 2](#). The basic argument for this consists essentially in the fact that a smooth solution of a PDE is also

a solution in the sense of distributions; this establishes (10). By density arguments it is clear that (10) also extends to $\varphi \in C_b$. Taking there $\varphi = 1$ for every ν_0 of Delta function type, we get (8).

An important point is constituted by the fact that this fundamental solution is also a transition probability, i.e. it verifies (13): this is the object of Problem 5, Chapter 1 of [13]. \square

7.2 Existence and uniqueness of a solution to a classical SDE

In this section we emphasize that we will state our results in two different contexts: in law, using Stroock-Varadhan ([28]) or Krylov ([17]) arguments for well-posedness of weak solutions of classical SDEs or Veretennikov ([30]) for strong solutions.

We fix a measurable function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$.

Lemma 27. *Let $\Phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be such that $a(t, x) = \Phi(t, x)\Phi(t, x)^t$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$. Let ν be a Borel probability measure on \mathbb{R}^d . Suppose the validity of Assumptions **A.1**, **A.2**, **A.3** and that $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded. Then the SDE*

$$\begin{cases} dX_t &= \Phi(t, X_t) dB_t + (b_0(t, X_t) + b(t, X_t, u(t, X_t))) dt, \\ X_0 &\sim \mathbf{u}_0. \end{cases} \quad (53)$$

where \mathbf{u}_0 is a probability measure admits weak existence and uniqueness in law (B being a Brownian motion).

Proof. We consider the martingale problem associated to

$$L_t = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \partial_{ij}^2 + \sum_{j=1}^d (b_{0,j}(t, x) + b_j(t, x, u(t, x))) \partial_j,$$

as introduced by Stroock and Varadhan in Chapter 6, Section 0 of [28]. We then note that Assumptions **A.1**, **A.2** and **A.3** are exactly the ones required for Theorem 7.2.1 in Chapter 7, Section 2 of [28], according to which our martingale problem is well-posed. We can then apply Corollary 3.4 (alongside with Proposition 3.5) in Chapter 5, Section 3 of [11], yielding that SDE (53) admits weak existence and uniqueness in law. \square

Remark 28. *Note that **A.3** is only required for the uniqueness in law of (53). If we drop **A.3** but assume instead that Φ is non-degenerate, we still have weak existence of a solution to (53), as can be inferred from Theorem 1 in Chapter 2.6. in [17] and the same arguments as in the proof of Lemma 27. Indeed it is possible to show that if a is nondegenerate, it is always possible to find a Φ such that $a = \Phi\Phi^t$ where Φ is nondegenerate.*

Under more restrictive assumptions, we have strong existence and pathwise uniqueness of a solution to (53).

Lemma 29. *Assume **A.1**, **A.2**, **B.3** and that $(t, x) \mapsto b(t, x, u(t, x))$ and $(t, x) \mapsto \Lambda(t, x, u(t, x))$ are bounded. Then (53) admits strong existence and pathwise uniqueness.*

Proof. The result follows from Theorem 1 in [30], because b is assumed to be bounded and u is assumed to be measurable. □

Remark 30.

- If $d = 1$, Theorem 1 in [30], admits some extensions, see Theorem 2 in [30].
- If $\Phi = I_d$ [30] result admits extensions to unbounded drifts, see [18].

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