

Forward Feynman-Kac type representation for semilinear nonconservative Partial Differential Equations

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Abstract

We propose a nonlinear forward Feynman-Kac type equation, which represents the solution of a non-conservative semilinear parabolic Partial Differential Equations (PDE). We show in particular existence and uniqueness. The solution of that type of equation can be approached via a weighted particle system.

Key words and phrases: Semilinear Partial Differential Equations; Nonlinear Feynman-Kac type functional; Particle systems; Probabilistic representation of PDEs.

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1 Introduction

In this paper we will focus on semilinear PDEs of the form

$$\begin{cases} \partial_t u = L_t^* u + u \Lambda(t, x, u, \nabla u) \\ u(0, \cdot) = \mathbf{u}_0, \end{cases} \quad (1.1)$$

where \mathbf{u}_0 is a Borel probability measure and L^* is a partial differential operator of the type

$$(L_t^* \varphi)(x) = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 ((\Phi \Phi^t)_{i,j}(t, x) \varphi)(x) - \sum_{i=1}^d \partial_i (g_i(t, x) \varphi)(x), \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^d), \quad (1.2)$$

with $\Phi : [0, T] \times \mathbb{R}^d \rightarrow M_{d,p}(\mathbb{R}^d)$, $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$. $u :]0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ will be the unknown function.

That equation is a particular case of nonlinear PDEs of the form

$$\begin{cases} \partial_t u = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 ((\Phi \Phi^t)_{i,j}(t, x, u) u) - \text{div}(g(t, x, u) u) + \Lambda(t, x, u, \nabla u) u, & \text{for any } t \in [0, T], \\ u(0, dx) = \mathbf{u}_0(dx); \end{cases} \quad (1.3)$$

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however this time $\Phi : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow M_{d,p}(\mathbb{R}^d)$, $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$. Allowing $\Lambda \neq 0$ encompasses the case of Burgers-Huxley or Burgers-Fisher equations which are of great importance to represent nonlinear phenomena in various fields such as biology [1, 27], physiology [19] and physics [33]. These equations have the particular interest to describe the interaction between the reaction mechanisms, convection effect, and diffusion transport. However our aim is also to consider (via time reversal) PDEs coming from stochastic control as non-linear HJB equations.

The underlying idea of our approach consists in extending, to fairly general non-conservative PDEs, the probabilistic representation of nonlinear Fokker-Planck equations which appears when $\Lambda = 0$. An interesting aspect of this strategy is that it is potentially able to represent an extended class of second order nonlinear PDEs. For (1.3), the target should be a forward probabilistic representation of the type

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, u(s, Y_s)) dW_s + \int_0^t g(s, Y_s, u(s, Y_s)) ds, & \text{with } Y_0 \sim \mathbf{u}_0, \\ u(t, \cdot) := \frac{d\nu_t}{dx} \quad \nu_t(\varphi) := \mathbb{E} \left[\varphi(Y_t) \exp \left\{ \int_0^t \Lambda(s, Y_s, u(s, Y_s), \nabla u(s, Y_s)) ds \right\} \right], & \varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}), t \in [0, T], \end{cases} \quad (1.4)$$

where W is a p -dimensional Brownian motion. (1.4) is a nonlinear stochastic differential equation (NLSDE) in the spirit of McKean, see e.g. [24]. The justification of the proposed probabilistic representation relies on the fact that, whenever a solution (Y, u) of (1.4) exists then u is a weak (in the sense of distributions) of (1.3); this follows in elementary way through an application of Itô formula.

When $\Lambda = 0$, several authors have studied NLSDEs of the form (1.4). Significant contributions are due to [32], [26], [25], in the case where the non linearity with respect to u are mollified in the diffusion and drift coefficients. In [17], the authors focused on the case when the coefficients depend pointwisely on u . The authors have proved strong existence and pathwise uniqueness of (1.4), when Φ and g are smooth and Lipschitz and Φ is non-degenerate. Other authors have more particularly studied an NLSDE of the form

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s)) dW_s, \quad Y_0 \sim \mathbf{u}_0, \\ u(t, x) dx \text{ is the law density of } Y_t, t > 0 \\ u(0, \cdot) = \mathbf{u}_0, \end{cases} \quad (1.5)$$

for which the particular case $d = 1$, $\Phi(u) = u^k$ for $k \geq 1$ was developed in [4]. When $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is only assumed to be bounded, measurable and monotone, existence/uniqueness results are still available in [6, 2]. A partial extension to the multidimensional case ($d \geq 2$) is exposed in [3].

The solutions of (1.3), when $\Lambda = 0$, are probability measures dynamics which often describe the macroscopic distribution law of a microscopic particle which behaves in a diffusive way. For that reason, those time evolution PDEs are conservative in the sense that their solutions $u(t, \cdot)$ verify the property $\int_{\mathbb{R}^d} u(t, x) dx$ to be constant in t (equal to 1, which is the mass of a probability measure). An interesting feature of this type of representation is that the law of the solution Y of the NLSDE can be characterized as the limiting empirical distribution of a large number of interacting particles. This is a consequence of the so called *propagation of chaos* phenomenon, already observed in the literature for the case of mollified dependence, see e.g. [18, 24, 32, 26] and [17] for the case of pointwise dependence. [9] has contributed to develop stochastic particle methods in the spirit of McKean to approach a PDE related to Burgers equation providing first the rate of convergence. Comparison with classical numerical analysis techniques was provided by [8].

As mentioned earlier, in this paper we will concentrate on the novelty constituted by the introduction of Λ depending on u and ∇u . For this step Φ, g will not depend on u . In this context the PDE (1.3) becomes

(1.1). For this case, (1.4) becomes

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s) dW_s + \int_0^t g(s, Y_s) ds, & \text{with } Y_0 \sim \mathbf{u}_0, \\ \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx = \mathbb{E} \left[\varphi(Y_t) \exp \left\{ \int_0^t \Lambda(s, Y_s, u(s, Y_s), \nabla u(s, Y_s)) ds \right\} \right], \end{cases} \quad \varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}), t \in [0, T]. \quad (1.6)$$

An alternative approach for representing this type of PDE is given by forward-backward stochastic differential equations. Those were initially developed in [29], see also [28] for a survey and [30] for a recent monograph on the subject. However, the extension of those equations to fully nonlinear PDEs still requires complex developments and is the subject of active research, see for instance [10]. Branching diffusion processes provide another probabilistic representation of semilinear PDEs, see e.g. [13, 15, 14]. Here again, extensions to second order nonlinear PDEs still constitutes a difficult issue.

As suggested, our method potentially allows to reach a certain significant class of PDEs with second-order non-linearity, if we allow the diffusion coefficient to also depend on u . The general framework where g and Φ also depend non linearly on u while Λ depends on u and ∇u has been partially investigated in [21], where the dependence of the coefficients with respect to u is mollified and Λ does not depend on ∇u . An associated interacting particle system converging to the solution of a regularized version of the nonlinear PDE has been proposed in [20], providing encouraging numerical performances. The originality of the present paper is to consider a pointwise dependence of Λ on both u and ∇u . The pointwise dependence on ∇u constitutes the major technical difficulty. For this we introduce a new approach based on the technique of *mild solutions* making use of the semigroupe associated with L_t . For this reason in this paper we concentrate on non-linearities only in Λ leaving extensions in the forthcoming paper [23] where we authorize the coefficient b to depend on u .

More specifically, we propose to associate (1.1) with a forward probabilistic representation given by a couple (Y, u) solution of (1.4) where Φ and g are the functions intervening in (1.2). In this case, the second line equation of (1.4) will be called **Feynman-Kac equation** and a solution $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ will be called **Feynman-Kac type representation** of (1.1). When Λ vanishes, the functions $(u(t, \cdot), t > 0)$ are indeed the marginal law densities of the process $(Y_t, t > 0)$ and (1.1) coincides with the classical Fokker-Planck PDE. When $\Lambda \neq 0$, the proof of well-posedness of the Feynman-Kac equation is not obvious and it is one of the contributions of the paper. The strategy used relies on two steps. Under a Lipschitz condition on Λ in Theorem 3.5, we first prove that a function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ (belonging to $L^1([0, T], W^{1,1}(\mathbb{R}^d))$) is a solution of the Feynman-Kac equation (1.4) if and only if it is a mild solution of (1.1). The latter concept is introduced in item 2. of Definition 2.1. Then, under Lipschitz type conditions on Φ and g , Theorem 3.6 establishes the existence and uniqueness of a mild solution of (1.1). As a second contribution, we propose and analyze a corresponding particle system. This relies on two approximation steps: a regularization procedure based on a kernel convolution and the law of large numbers. The convergence of the particle system is stated in Theorem 5.3.

The theoretical analysis of the performance of the time-discretized algorithm related to the present paper has been performed in Theorem 3.4 in [22]. In that paper we test the algorithm with respect to the Burgers and KPZ equations for which there are explicit solutions.

2 Preliminaries

2.1 Notations

Let $d \in \mathbb{N}^*$. Let us consider $\mathcal{C}^d := \mathcal{C}([0, T], \mathbb{R}^d)$ metricized by the supremum norm $\|\cdot\|_\infty$, equipped with its Borel σ -field $\mathcal{B}(\mathcal{C}^d)$ and endowed with the topology of uniform convergence. If (E, d_E) is a Polish space, $\mathcal{P}(E)$ denotes the Polish space (with respect to the weak convergence topology) of Borel probability measures on E naturally equipped with its Borel σ -field $\mathcal{B}(\mathcal{P}(E))$. The reader can consult Proposition 7.20 and Proposition 7.23, Section 7.4 Chapter 7 in [5] for more exhaustive information. When $d = 1$, we often omit it and we simply note $\mathcal{C} := \mathcal{C}^1$. $\mathcal{C}_b(E)$ denotes the space of bounded, continuous real-valued functions on E .

In this paper, \mathbb{R}^d is equipped with the Euclidean scalar product \cdot and $|x|$ stands for the induced norm for $x \in \mathbb{R}^d$. The gradient operator for functions defined on \mathbb{R}^d is denoted by ∇ . If a function u depends on a variable $x \in \mathbb{R}^d$ and other variables, we still denote by ∇u the gradient of u with respect to x , if there is no ambiguity. $M_{d,p}(\mathbb{R})$ denotes the space of $\mathbb{R}^{d \times p}$ real matrices equipped with the Frobenius norm (also denoted $|\cdot|$), i.e. the one induced by the scalar product $(A, B) \in M_{d,p}(\mathbb{R}^d) \times M_{d,p}(\mathbb{R}) \mapsto \text{Tr}(A^t B)$ where A^t stands for the transpose matrix of A and Tr is the trace operator. \mathcal{S}_d is the set of symmetric, non-negative definite $d \times d$ real matrices and \mathcal{S}_d^+ the set of strictly positive definite matrices of \mathcal{S}_d .

$\mathcal{M}_f(\mathbb{R}^d)$ is the space of finite Borel measures on \mathbb{R}^d . When it is endowed with the weak convergence topology, $\mathcal{B}(\mathcal{M}_f(\mathbb{R}^d))$ stands for its Borel σ -field. It is well-known that $(\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV})$ is a Banach space, where $\|\cdot\|_{TV}$ denotes the total variation norm. $\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz fast decreasing test functions and $\mathcal{S}'(\mathbb{R}^d)$ is its dual. $\mathcal{C}_b(\mathbb{R}^d)$ is the space of bounded, continuous functions on \mathbb{R}^d and $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the space of smooth functions with compact support. For any positive integers $p, k \in \mathbb{N}$, $\mathcal{C}_b^{k,p} := \mathcal{C}_b^{k,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$ denotes the set of continuously differentiable bounded functions $[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ with uniformly bounded derivatives with respect to the time variable t (resp. with respect to space variable x) up to order k (resp. up to order p). In particular, for $k = p = 0$, $\mathcal{C}_b^{0,0}$ coincides with the space of bounded, continuous functions also denoted by \mathcal{C}_b . $\mathcal{C}_b^\infty(\mathbb{R}^d)$ is the space of bounded and smooth functions. $\mathcal{C}_0(\mathbb{R}^d)$ denotes the space of continuous functions with compact support in \mathbb{R}^d . For $r \in \mathbb{N}$, $W^{r,p}(\mathbb{R}^d)$ is the Sobolev space of order r in $(L^p(\mathbb{R}^d), \|\cdot\|_p)$, with $1 \leq p \leq \infty$.

For convenience we introduce the following notation.

- $V : [0, T] \times \mathcal{C}^d \times \mathcal{C} \times \mathcal{C}^d$ is defined for any functions $x \in \mathcal{C}^d, y \in \mathcal{C}$ and $z \in \mathcal{C}^d$, by

$$V_t(x, y, z) := \exp\left(\int_0^t \Lambda(s, x_s, y_s, z_s) ds\right) \quad \text{for any } t \in [0, T]. \quad (2.1)$$

The finite increments theorem gives, for all $(a, b) \in \mathbb{R}^2$, we have

$$\exp(a) - \exp(b) = (b - a) \int_0^1 \exp(\alpha a + (1 - \alpha)b) d\alpha. \quad (2.2)$$

Therefore, if Λ is supposed to be bounded and Lipschitz w.r.t. to its space variables (x, y, z) , uniformly w.r.t. t , we observe that (2.2) implies that, for all $t \in [0, T], x, x' \in \mathcal{C}^d, y, y' \in \mathcal{C}, z, z' \in \mathcal{C}^d$,

$$|V_t(x, y, z) - V_t(x', y', z')| \leq L_\Lambda e^{tM_\Lambda} \int_0^t (|x_s - x'_s| + |y_s - y'_s| + |z_s - z'_s|) ds, \quad (2.3)$$

M_Λ (resp. L_Λ) denoting an upper bound of $|\Lambda|$ (resp. the Lipschitz constant of Λ), see also Assumption 2.

- For every ε , $K_\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ denotes a mollifier such that

$$K_\varepsilon(x) := \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}\right), \forall x \in \mathbb{R}^d, \quad (2.4)$$

where K is a probability density on \mathbb{R}^d such that

$$K \in W^{1,1}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d). \quad (2.5)$$

In the sequel, K may be asked to additionally verify the following conditions.

$$\kappa := \frac{1}{2} \int_{\mathbb{R}^d} |x| K(x) dx < \infty. \quad (2.6)$$

$$\int_{\mathbb{R}^d} |x|^{d+1} K(x) dx < \infty, \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^{d+1} |\nabla K(x)| dx < \infty. \quad (2.7)$$

In the whole paper, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ will denote a filtered probability space and W an \mathbb{R}^p -valued (\mathcal{F}_t) -Brownian motion.

2.2 Mild and Weak solutions

We first introduce the following assumption.

Assumption 1. 1. Φ and g are functions defined on $[0, T] \times \mathbb{R}^d$ taking values in $M_{d,p}(\mathbb{R}^d)$ and \mathbb{R}^d . There exist $L_\Phi, L_g > 0$ such that for any $t \in [0, T]$, $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} |\Phi(t, x) - \Phi(t, x')| &\leq L_\Phi |x - x'|, \\ |g(t, x) - g(t, x')| &\leq L_g |x - x'|. \end{aligned}$$

2. The functions $s \in [0, T] \mapsto |\Phi(s, 0)|$ and $s \in [0, T] \mapsto |g(s, 0)|$ are bounded.

Given any $\sigma(W_r, r \leq s)$ -measurable r.v. Y_s , classical theorems for SDE with Lipschitz coefficients imply strong existence and pathwise uniqueness for the SDE

$$dY_t = \Phi(t, Y_t) dW_t + g(t, Y_t) dt, t \in [s, T]. \quad (2.8)$$

Section 2.2, Chapter 2 in [31] and Section 2.1, Chapter 1 of [12] one introduces the notion of **Markov transition function**. Under Assumption 1, by Theorem 3.1 chap. 5 of [12], it is well-known, that there exists a good family of Markov transition functions $P(s, x_0, t, \cdot)$ such that, for every $0 \leq s \leq t \leq T$ and Borel subset A of \mathbb{R}^d we have

$$P\{Y_t \in A | \sigma(W_r, r \leq s)\} = P\{Y_t \in A | Y_s\} = P(s, Y_s, t, A). \quad (2.9)$$

Let $s = 0$, $Y_0 \sim \mathbf{u}_0$. From now on Y will be the unique strong solution of the SDE

$$dY_t = \Phi(t, Y_t) dW_t + g(t, Y_t) dt, t \in [0, T]. \quad (2.10)$$

For $t \in [0, T]$, the marginal law of Y_t is given for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ by

$$\mathbb{E}[\varphi(Y_t)] = \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) \int_{\mathbb{R}^d} \varphi(x) P(0, x_0, t, dx). \quad (2.11)$$

In the whole paper we will write $a = \Phi \Phi^t$; in particular $a : [0, T] \times \mathbb{R}^d \rightarrow S_d$. Through some definitions, we

make here precise in which sense we will consider solutions of the PDE (1.1). We are interested in different concepts of solutions $u : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ of that semilinear PDE where, for $t \in [0, T]$, L_t is given by

$$(L_t \varphi)(x) = \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, x) \partial_{ij}^2 \varphi(x) + \sum_{i=1}^d g_i(t, x) \partial_i \varphi(x), \quad \varphi \in C_0^\infty(\mathbb{R}^d). \quad (2.12)$$

Its "adjoint" L_t^* defined in (1.2), verifies

$$\int_{\mathbb{R}^d} L_t \varphi(x) \psi(x) dx = \int_{\mathbb{R}^d} \varphi(x) L_t^* \psi(x) dx, \quad (\varphi, \psi) \in C_0^\infty(\mathbb{R}^d), t \in [0, T]. \quad (2.13)$$

Let ν_0 be a Borel probability measure on \mathbb{R}^d . By an easy application of Itô formula to Y_t when $Y_s \sim \nu_0$ with smooth φ with compact support, one can show that the measure-valued function

$$\nu_s(t, dx) := \int_{\mathbb{R}^d} P(s, x_0, t, dx) \nu_0(dx_0) \quad (2.14)$$

is a solution in the sense of distributions to the Fokker-Planck equation

$$\begin{cases} \partial_t \nu_s(t, dx) = L_t^* \nu_s(t, dx) & \forall (t, x) \in]s, T] \times \mathbb{R}^d \\ \nu_s(s, \cdot) = \nu_0, \end{cases} \quad (2.15)$$

i.e., for all $\varphi \in C_0^\infty$,

$$\int_{\mathbb{R}^d} \varphi(x) \nu_s(t, dx) - \int_{\mathbb{R}^d} \varphi(x) \nu_0(dx) = \int_s^t \int_{\mathbb{R}^d} L_r \varphi(x) \nu_s(r, dx) dr. \quad (2.16)$$

In particular (2.15) with $\nu_0 = \delta_{x_0}$ says that

$$\int_{\mathbb{R}^d} \varphi(x) P(s, x_0, t, dx) - \varphi(x_0) = \int_s^t \int_{\mathbb{R}^d} L_r \varphi(x) P(s, x_0, r, dx) dr, \quad (2.17)$$

which means

$$\begin{cases} \partial_t P(s, x_0, t, \cdot) = L_t^* P(s, x_0, t, \cdot) \\ P(s, x_0, s, \cdot) = \delta_{x_0}, \quad 0 \leq s \leq T, x_0 \in \mathbb{R}^d. \end{cases} \quad (2.18)$$

Let $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be bounded, Borel measurable, we recall the notions of **weak solution** and **mild solution** associated to (1.1).

Definition 2.1. Let $u : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a Borel function such that for every $t \in]0, T]$, $u(t, \cdot) \in W^{1,1}(\mathbb{R}^d)$.

1. u will be called **weak solution** of (1.1) if for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) u(t, x) dx - \int_{\mathbb{R}^d} \varphi(x) \mathbf{u}_0(dx) &= \int_0^t \int_{\mathbb{R}^d} u(s, x) L_s \varphi(x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \varphi(x) \Lambda(s, x, u(s, x), \nabla u(s, x)) u(s, x) dx ds. \end{aligned} \quad (2.19)$$

2. u will be called **mild solution** of (1.1) if for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) u(t, x) dx &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) P(0, x_0, t, dx) \\ &+ \int_{[0,t] \times \mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(x) P(s, x_0, t, dx) \right) \Lambda(s, x_0, u(s, x_0), \nabla u(s, x_0)) u(s, x_0) dx_0 ds. \end{aligned} \quad (2.20)$$

As mentioned in the introduction, a natural approach to show the link between (1.1) and (1.6) consists in applying Itô's formula to the solution Y of (2.10): if (Y, u) is a solution of (1.6), then u is a weak solution of (1.1). However, in this paper, instead of the notion of weak solution, we will make use of the notion of mild solution. The link between those two notions is discussed in the proposition below.

Proposition 2.2. *We assume that $\nu = 0$ is the unique solution in the sense of distributions of (2.15) with $\nu_0 = 0$. Then, u is a mild solution of (1.1) if and only if u is a weak solution of (1.1)*

Proof. Postponed to the Appendix, see Section 6.3. □

Remark 2.3. *There exist several sets of technical assumptions (see e.g. [7, 12]) leading to the uniqueness assumed in Proposition 2.2 above. In particular, under items 1., 2. and 3. of Assumption 2 stated in Section 3 (which will constitutes our framework in the sequel), Theorem 4.7 in Chapter 4 of [12] ensures (classical) existence and uniqueness of the solution of (2.15), see also Lemma 6.4 in the Appendix.*

3 Feynman-Kac type representation

We suppose here the validity of Assumption 1. Let $\mathbf{u}_0 \in \mathcal{P}(\mathbb{R}^d)$ and fix a random variable Y_0 distributed according to \mathbf{u}_0 and consider the strong solution Y of (2.10). From now on Y will be fixed.

The aim of this section is to show how a mild solution of (1.1) can be linked with a Feynman-Kac type equation, where we recall that a solution is given by a function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the second line equation of (1.4).

Given $\tilde{\Lambda} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ a bounded, Borel measurable function, let us consider the measure-valued map $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ defined by

$$\int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) = \mathbb{E} \left[\varphi(Y_t) \exp \left(\int_0^t \tilde{\Lambda}(s, Y_s) ds \right) \right], \text{ for all } \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0, T]. \quad (3.1)$$

The first proposition below shows how the map $t \mapsto \mu(t, \cdot)$ can be characterized as a solution of the linear parabolic PDE

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}(t, x) v \\ v(0, \cdot) = \mathbf{u}_0. \end{cases} \quad (3.2)$$

Before stating the corresponding proposition, we introduce the notion of *measure-mild solution*.

Definition 3.1. *Let $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ be a measure-valued map such that*

$$\int_0^T \|\mu(t, \cdot)\|_{TV} dt < \infty.$$

*μ will be called **measure-mild solution** of (3.2) if for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$,*

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) P(0, x_0, t, dx) \\ &+ \int_{[0, t] \times \mathbb{R}^d} \left(\int_{\mathbb{R}^d} P(r, x_0, t, dx) \varphi(x) \right) \tilde{\Lambda}(r, x_0) \mu(r, dx_0) dr. \end{aligned} \quad (3.3)$$

Remark 3.2. *1. By usual approximation arguments, it is not difficult to show that an equivalent formulation for Definition 2.1 can be expressed taking φ in $\mathcal{C}_b(\mathbb{R}^d)$ instead of $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.*

2. Although the definition of **mild solution** (see item 2. of Definition of 2.1) and the one of **measure-mild solution** seem to be formally close, the two concepts do not make sense in the same situations. Indeed, the notion of mild-solution makes sense for PDEs with nonlinear terms of the general form $\Lambda(t, x, u, \nabla u)$, whereas a measure-mild solution can exist only for linear PDEs. However, in the case where a measure μ on \mathbb{R}^d , absolutely continuous w.r.t. the Lebesgue measure dx , is a measure-mild solution of the linear PDE (3.2), its density indeed coincides with the mild solution (in the sense of item 2. of Definition 2.1) of (3.2).

Proposition 3.3. Under Assumption 1 the measure-valued map μ defined by (3.1) is the unique measure-mild solution of

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}(t, x)v \\ v(0, \cdot) = \mathbf{u}_0, \end{cases} \quad (3.4)$$

where the operator L_t^* is defined by (1.2).

Proof. We first prove that a function μ defined by (3.1) is a measure-mild solution of (3.4).

Observe that for all $t \in [0, T]$,

$$\exp\left(\int_0^t \tilde{\Lambda}(r, Y_r) dr\right) = 1 + \int_0^t \tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds} dr. \quad (3.5)$$

From (3.1), it follows that for all test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $t \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) &= \mathbb{E}\left[\varphi(Y_t) \exp\left(\int_0^t \tilde{\Lambda}(r, Y_r) dr\right)\right] \\ &= \mathbb{E}[\varphi(Y_t)] + \int_0^t \mathbb{E}\left[\varphi(Y_t) \tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds}\right] dr. \end{aligned} \quad (3.6)$$

On the one hand, by (2.11), we have

$$\mathbb{E}[\varphi(Y_t)] = \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) \int_{\mathbb{R}^d} \varphi(x) P(0, x_0, t, dx), \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \text{ and } t \in [0, T]. \quad (3.7)$$

On the other hand, using (2.9) yields, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $0 \leq r \leq t$,

$$\begin{aligned} \mathbb{E}\left[\varphi(Y_t) \tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds}\right] &= \mathbb{E}\left[\tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds} \mathbb{E}[\varphi(Y_t) | Y_r]\right] \\ &= \mathbb{E}\left[\left(\tilde{\Lambda}(r, Y_r) \int_{\mathbb{R}^d} \varphi(x) P(r, Y_r, t, dx)\right) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds}\right] \\ &= \int_{\mathbb{R}^d} \left(\tilde{\Lambda}(r, x_0) \int_{\mathbb{R}^d} \varphi(x) P(r, x_0, t, dx)\right) \mu(r, dx_0), \end{aligned} \quad (3.8)$$

where the third equality above comes from (3.1) applied to the bounded, measurable test function $z \mapsto \tilde{\Lambda}(r, z) \int_{\mathbb{R}^d} \varphi(x) P(r, z, t, dx)$. Injecting (3.8) and (3.7) in the right-hand side (r.h.s.) of (3.6) gives for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) &= \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) \int_{\mathbb{R}^d} P(0, x_0, t, dx) \varphi(x) dx \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \mu(r, dx_0) \tilde{\Lambda}(r, x_0) \int_{\mathbb{R}^d} P(r, x_0, t, dx) \varphi(x) dr. \end{aligned} \quad (3.9)$$

It remains now to prove uniqueness of the measure-mild solution of (3.4). We recall that $\mathcal{M}_f(\mathbb{R}^d)$ denotes the vector space of finite Borel measures on \mathbb{R}^d , that is here equipped with the total variation norm $\|\cdot\|_{TV}$. We also recall that an equivalent definition of the total variation norm is given by

$$\|\mu\|_{TV} = \sup_{\substack{\psi \in \mathcal{C}_b(\mathbb{R}^d) \\ \|\psi\|_\infty \leq 1}} \left| \int_{\mathbb{R}^d} \psi(x) \mu(dx) \right|. \quad (3.10)$$

Consider $t \in [0, T]$ and let μ_1, μ_2 be two measure-mild solutions of PDE (3.4). We set $\nu := \mu_1 - \mu_2$. Since $\tilde{\Lambda}$ is bounded, we observe that (3.1) implies $\|\nu(t, \cdot)\|_{TV} < +\infty$. Moreover, taking into account item 1. of Remark 3.2, we have that ν satisfies,

$$\forall \varphi \in \mathcal{C}_b(\mathbb{R}^d), \int_{\mathbb{R}^d} \varphi(x) \nu(t, dx) = \int_0^t \int_{\mathbb{R}^d} \tilde{\Lambda}(r, x_0) \nu(r, dx_0) \int_{\mathbb{R}^d} \varphi(x) P(r, x_0, t, dx) dr. \quad (3.11)$$

Taking the supremum over φ such that $\|\varphi\|_\infty \leq 1$ in each side of (3.11), we get

$$\|\nu(t, \cdot)\|_{TV} \leq \sup_{(s, x) \in [0, T] \times \mathbb{R}^d} |\tilde{\Lambda}(s, x)| \int_0^t \|\nu(r, \cdot)\|_{TV} dr. \quad (3.12)$$

Gronwall's lemma implies that $\nu(t, \cdot) = 0$. Uniqueness of measure-mild solution for (3.4) follows. This ends the proof. \square

The next lemma shows how a measure-mild solution of (3.4), which is a function defined on $[0, T]$ can be built by defining it recursively on each sub-interval of the form $[r, r + \tau]$. In particular, it will be used in Theorem 3.6 and Proposition 4.4. Its proof is postponed in Appendix (see Section 6.4).

Lemma 3.4. *Let N be a strictly positive integer. Let us fix $\tau > 0$ be a real constant and $\delta := (\alpha_0 := 0 < \dots < \alpha_k := k\tau < \dots < \alpha_N := T)$ be a finite partition of $[0, T]$.*

A measure-valued map $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ satisfies

$$\begin{cases} \mu(0, \cdot) = \mathbf{u}_0 \\ \mu(t, dx) = \int_{\mathbb{R}^d} P(k\tau, x_0, t, dx) \mu(k\tau, dx_0) + \int_{k\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0), \end{cases} \quad (3.13)$$

for all $t \in [k\tau, (k+1)\tau]$ and $k \in \{0, \dots, N-1\}$, if and only if μ is a measure-mild solution (in the sense of Definition 3.1) of (3.4).

We now come back to the case where the bounded, Borel measurable real-valued function Λ is defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$. Let $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ belonging to $L^1([0, T], W^{1,1}(\mathbb{R}^d))$. In the sequel, we set $\tilde{\Lambda}^u(t, x) := \Lambda(t, x, u(t, x), \nabla u(t, x))$. μ^u will denote the measure-valued map μ defined by (3.1) with $\tilde{\Lambda} = \tilde{\Lambda}^u$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \mu^u(t, dx) = \mathbb{E} \left[\varphi(Y_t) \exp \left(\int_0^t \tilde{\Lambda}^u(s, Y_s) ds \right) \right], \text{ for all } \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0, T]. \quad (3.14)$$

By Proposition 3.3, it follows that μ^u is the unique measure-mild solution of the linear PDE (3.4) with $\tilde{\Lambda} = \tilde{\Lambda}^u$. (3.14) can be interpreted as a Feynman-Kac type representation for the measure-mild solution μ^u of the linear PDE (3.4), for the corresponding $\tilde{\Lambda}^u$. More generally, Theorem 3.5 below establishes such representation formula for a mild solution of the semilinear PDE (1.1).

Theorem 3.5. *Assume that Assumption 1 is fulfilled. We indicate by Y the unique strong solution of (2.10). Suppose that $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded and Borel measurable. A function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ in $L^1([0, T], W^{1,1}(\mathbb{R}^d))$ is a mild solution of (1.1) if and only if, for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$, $t \in [0, T]$,*

$$\int_{\mathbb{R}^d} \varphi(x) u(t, x) dx = \mathbb{E} \left[\varphi(Y_t) \exp \left(\int_0^t \Lambda(s, Y_s, u(s, Y_s), \nabla u(s, Y_s)) ds \right) \right]. \quad (3.15)$$

A function u verifying (3.15) will be called a **Feynman-Kac type representation** of (1.1).

Proof. We set

$$\tilde{\Lambda}^u(t, x) := \Lambda(t, x, u(t, x), \nabla u(t, x)), \quad (3.16)$$

which is bounded and Borel measurable. The result follows by applying Proposition 3.3 with $\tilde{\Lambda} = \tilde{\Lambda}^u$. \square

We now precise more restrictive assumptions to ensure regularity properties of the transition probability function $P(s, x_0, t, dx)$ used in the sequel.

Assumption 2. 1. Φ and g are functions defined on $[0, T] \times \mathbb{R}^d$ taking values respectively in $M_{d,p}(\mathbb{R})$ and \mathbb{R}^d . There exist $\alpha \in]0, 1]$, $C_\alpha, L_\Phi, L_g > 0$, such that for any $(t, t', x, x') \in [0, T] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} |\Phi(t, x) - \Phi(t', x')| &\leq C_\alpha |t - t'|^\alpha + L_\Phi |x - x'|, \\ |g(t, x) - g(t', x')| &\leq C_\alpha |t - t'|^\alpha + L_g |x - x'|. \end{aligned}$$

2. Φ and g belong to $C_b^{0,3}$. In particular, Φ, g are uniformly bounded and M_Φ (resp. M_g) denote the upper bound of $|\Phi|$ (resp. $|g|$).

3. Φ is non-degenerate, i.e. there exists $c > 0$ such that for all $x \in \mathbb{R}^d$

$$\inf_{s \in [0, T]} \inf_{v \in \mathbb{R}^d \setminus \{0\}} \frac{\langle v, \Phi(s, x) \Phi^t(s, x) v \rangle}{|v|^2} \geq c > 0. \quad (3.17)$$

4. Λ is a Borel real-valued function defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ and Lipschitz uniformly w.r.t. (t, x) i.e. there exists a finite positive real, L_Λ , such that for any $(t, x, z_1, z'_1, z_2, z'_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$, we have

$$|\Lambda(t, x, z_1, z_2) - \Lambda(t, x, z'_1, z'_2)| \leq L_\Lambda (|z_1 - z'_1| + |z_2 - z'_2|). \quad (3.18)$$

5. Λ is supposed to be uniformly bounded: let M_Λ be an upper bound for $|\Lambda|$.

6. \mathbf{u}_0 is a Borel probability measure on \mathbb{R}^d admitting a bounded density (still denoted by the same letter) belonging to $W^{1,1}(\mathbb{R}^d)$.

Theorem 3.6. Under Assumption 2, there exists a unique mild solution u of (1.1) in $L^1([0, T], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$.

The idea of the proof will be first to construct a unique "mild solution" u^k of (1.1) on each subintervals of the form $[k\tau, (k+1)\tau]$ with $k \in \{0, \dots, N-1\}$ and $\tau > 0$ a constant supposed to be fixed for the moment. This will be the object of Lemma 3.7. Secondly we will show that the function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, defined by being equal to u^k on each $[k\tau, (k+1)\tau]$, is indeed a mild solution of (1.1) on $[0, T] \times \mathbb{R}^d$. This will be a consequence of Lemma 3.4. Finally, uniqueness will follow classically from the Lipschitz property of Λ .

By Assumption 2 and item 1. of Lemma 6.4, the transition kernels are absolutely continuous and $P(s, x_0, t, dx) = p(s, x_0, t, x) dx$ for some Borel function p . Let us fix $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. For $r \in [0, T - \tau]$, we first define a function \widehat{u}_0 on $[r, r + \tau] \times \mathbb{R}^d$ by setting

$$\widehat{u}_0(r, \phi)(t, x) := \int_{\mathbb{R}^d} p(r, x_0, t, x) \phi(x_0) dx_0, \quad (t, x) \in [r, r + \tau] \times \mathbb{R}^d. \quad (3.19)$$

Consider now the map $\Pi : L^1([r, r + \tau], W^{1,1}(\mathbb{R}^d)) \rightarrow L^1([r, r + \tau], W^{1,1}(\mathbb{R}^d))$ given by

$$\Pi(v)(t, x) := \int_r^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) \Lambda(s, x_0, v + \widehat{u}_0(r, \phi), \nabla(v + \widehat{u}_0(r, \phi)))(v + \widehat{u}_0)(r, \phi)(s, x_0) dx_0, \quad (3.20)$$

$$\Lambda(t, z, v, \nabla v) := \Lambda(t, z, v(t, z), \nabla v(t, z)) \quad \text{with} \quad (t, z) \in [0, T] \times \mathbb{R}^d, \quad (3.21)$$

that will also be used in the sequel.

Later, the dependence on r, ϕ will be omitted when it is self-explanatory. Since ϕ belongs to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, also taking into account (6.16), we have

$$\|\widehat{u}_0(t, \cdot)\|_1 \leq \|\phi\|_1 \quad \text{and} \quad \|\widehat{u}_0(t, \cdot)\|_\infty \leq \|\phi\|_\infty, \quad \text{if } t \in [r, r + \tau]. \quad (3.22)$$

The lemma below establishes, under a suitable choice of $\tau > 0$, existence and uniqueness of the mild solution on $[r, r + \tau]$, with initial condition ϕ at time r , i.e. existence and uniqueness of the fixed-point for the application Π .

Lemma 3.7. *Assume the validity of Assumption 2. Let $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.*

Let $M > 0$ such that $M \geq \max(\|\phi\|_\infty; \|\phi\|_1)$. Then, there is $\tau > 0$ only depending on M_Λ and on C_u, c_u (the constants coming from inequalities (6.14) and (6.15), only depending on Φ, g) such that for any $r \in [0, T - \tau]$, Π admits a unique fixed-point in $L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$, where $B(0, M)$ (resp. $B_\infty(0, M)$) denotes the centered ball in $W^{1,1}(\mathbb{R}^d)$ (resp. $L^\infty([r, r + \tau] \times \mathbb{R}^d, \mathbb{R})$) of radius M .

Proof. We first insist on the fact that all along the proof, the dependence of \widehat{u}_0 w.r.t. r, ϕ in (3.20) will be omitted to simplify notations. Let us fix $r \in [0, T - \tau]$.

The rest of the proof relies on a fixed-point argument in the Banach space $L^1([r, r + \tau], W^{1,1}(\mathbb{R}^d))$ equipped with the norm $\|f\|_{1,1} := \int_r^{r+\tau} \|f(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds$ and for the map Π (3.20). Moreover, we emphasize that $L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$ is complete as a closed subset of $L^1([r, r + \tau], B(0, M))$.

We first check that $\Pi(L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)) \subset L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$. Let us fix $v \in L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$. For $t \in [r, r + \tau]$,

$$\begin{aligned} \|\Pi(v)(t, \cdot)\|_1 &= \int_{\mathbb{R}^d} |\Pi(v)(t, x)| dx \\ &\leq M_\Lambda \int_r^t (\|v(s, \cdot)\|_1 + \|\widehat{u}_0(s, \cdot)\|_1) ds \\ &\leq 2M_\Lambda M \tau, \end{aligned} \quad (3.23)$$

where we have used the fact that $x \mapsto p(s, x_0, t, x)$ is a probability density, the boundedness of Λ and the bounds $\|v(s, \cdot)\|_1 \leq M$ and $\|\widehat{u}_0(s, \cdot)\|_1 \leq M$ for $s \in [r, r + \tau]$.

Let us fix $t \in [r, r + \tau]$. By item 2. of Lemma 6.4, taking into account inequality (6.15), we differentiate under the integral sign with respect to x , to obtain that $\nabla \Pi(v)(t, \cdot)$ exists (in the sense of distributions) and is a real-valued function such that for almost all $x \in \mathbb{R}^d$,

$$\nabla \Pi(v)(t, x) = \int_r^t ds \int_{\mathbb{R}^d} \nabla_x p(s, x_0, t, x) (v + \widehat{u}_0)(s, x_0) \Lambda(s, x_0, v + \widehat{u}_0, \nabla(v + \widehat{u}_0)) dx_0. \quad (3.24)$$

Integrating each side of (3.24) on \mathbb{R}^d w.r.t. dx and using inequality (6.15) yield

$$\begin{aligned} \|\nabla \Pi(v)(t, \cdot)\|_1 &= \int_{\mathbb{R}^d} |\nabla \Pi(v)(t, x)| dx \\ &\leq M_\Lambda \int_r^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} C_u q(s, x_0, t, x) (|v(s, x_0)| + |\widehat{u}_0(s, x_0)|) dx_0 \\ &= C_u M_\Lambda \int_r^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}^d} (|v(s, x_0)| + |\widehat{u}_0(s, x_0)|) dx_0 \\ &\leq C_u M_\Lambda \int_r^t (\|v(s, \cdot)\|_1 + \|\widehat{u}_0(s, \cdot)\|_1) \frac{ds}{\sqrt{t-s}} \\ &\leq 4C_u M_\Lambda M \sqrt{\tau}, \end{aligned} \quad (3.25)$$

where the constant C_u and the Gaussian kernel q come from inequality (6.15) and only depending on Φ and g . Consequently, taking into account (3.23) and (3.25), we obtain,

$$\|\Pi(v)\|_{1,1} = \int_r^{r+\tau} \|\Pi(v)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \leq 2MM_\Lambda(\tau^2 + 2C_u\tau\sqrt{\tau}). \quad (3.26)$$

Moreover using inequality (6.14), gives

$$\|\Pi(v)\|_\infty \leq 2C_uMM_\Lambda\tau. \quad (3.27)$$

Now, setting

$$\tau := \min\left(\sqrt{\frac{1}{6M_\Lambda}}; \left(\frac{1}{6C_uM_\Lambda}\right)^{\frac{2}{3}}; \frac{1}{2C_uM_\Lambda}\right), \quad (3.28)$$

we have

$$2MM_\Lambda(\tau^2 + 2C_u\tau\sqrt{\tau}) \leq M \quad \text{and} \quad 2C_uMM_\Lambda\tau \leq M,$$

which implies

$$\|\Pi(v)\|_{1,1} \leq M \quad \text{and} \quad \|\Pi(v)\|_\infty \leq M.$$

We deduce that $\Pi(v) \in L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$.

Let us fix $t \in [r, r + \tau]$, $v_1, v_2 \in L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$. Λ being bounded and Lipschitz, the notation introduced in (3.21) and inequality (2.3) imply

$$\begin{aligned} \|\Pi(v_1)(t, \cdot) - \Pi(v_2)(t, \cdot)\|_1 &\leq \int_r^t ds \int_{\mathbb{R}^d} \left| v_1(s, x_0)\Lambda(s, x_0, v_1 + \widehat{u}_0, \nabla(v_1 + \widehat{u}_0)) - v_2(s, x_0)\Lambda(s, x_0, v_2 + \widehat{u}_0, \nabla(v_2 + \widehat{u}_0)) \right| dx_0 \\ &\quad + \int_r^t ds \int_{\mathbb{R}^d} |\widehat{u}_0(s, x_0)| \left| \Lambda(s, x_0, v_1 + \widehat{u}_0, \nabla(v_1 + \widehat{u}_0)) - \Lambda(s, x_0, v_2 + \widehat{u}_0, \nabla(v_2 + \widehat{u}_0)) \right| dx_0 \\ &\leq \int_r^t ds \left(\int_{\mathbb{R}^d} |v_1(s, x_0) - v_2(s, x_0)| |\Lambda(s, x_0, v_1 + \widehat{u}_0, \nabla(v_1 + \widehat{u}_0))| dx_0 \right. \\ &\quad \left. + L_\Lambda \int_r^t ds \int_{\mathbb{R}^d} (|\widehat{u}_0(s, x_0)| + |v_2(s, x_0)|) |v_1(s, x_0) - v_2(s, x_0)| dx_0 \right. \\ &\quad \left. + L_\Lambda \int_r^t ds \int_{\mathbb{R}^d} (|\widehat{u}_0(s, x_0)| + |v_2(s, x_0)|) |\nabla v_1(s, x_0) - \nabla v_2(s, x_0)| dx_0 \right) \\ &\leq (M_\Lambda + 2ML_\Lambda) \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds, \end{aligned} \quad (3.29)$$

where we have used the fact that $\int_{\mathbb{R}^d} p(s, x_0, t, x) dx = 1$, $0 \leq s < t \leq T$.

In the same way using inequality (6.15)

$$\begin{aligned} \left\| \nabla \left(\Pi(v_1) - \Pi(v_2) \right) (t, \cdot) \right\|_1 &\leq C_u(M_\Lambda + 2ML_\Lambda) \int_{\mathbb{R}^d} \int_r^t \int_{\mathbb{R}^d} \frac{1}{\sqrt{t-s}} q(s, x_0, t, x) \left(|v_1(s, x_0) - v_2(s, x_0)| \right. \\ &\quad \left. + |\nabla v_1(s, x_0) - \nabla v_2(s, x_0)| \right) dx_0 ds dx. \end{aligned} \quad (3.30)$$

By Fubini's theorem we have

$$\left\| \nabla \left(\Pi(v_1) - \Pi(v_2) \right) (t, \cdot) \right\|_1 \leq \tilde{C} \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds, \quad (3.31)$$

with $\tilde{C} := \tilde{C}(C_u, c_u, M_\Lambda, L_\Lambda, M)$ some positive constant. From (3.29) and (3.31), we deduce there exists a strictly positive constant $C = C(C_u, c_u, \Phi, g, \Lambda, M)$ (which may change from line to line) such that for all

$t \in [r, r + \tau]$,

$$\begin{aligned} \|\Pi(v_1)(t, \cdot) - \Pi(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} &\leq C \left\{ \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds \right. \\ &\quad \left. + \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds \right\}. \end{aligned} \quad (3.32)$$

Iterating the procedure once again yields

$$\begin{aligned} \|\Pi^2(v_1)(t, \cdot) - \Pi^2(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} &\leq C \left\{ \int_r^t \int_r^s \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds \right. \\ &\quad \left. + \int_r^t \int_r^s \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds \right\}, \end{aligned} \quad (3.33)$$

for all $t \in [r, r + \tau]$. Interchanging the order in the second integral in the r.h.s. of (3.33), we obtain

$$\begin{aligned} \int_r^t \int_r^s \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds &= \int_r^t d\theta \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \int_\theta^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} ds \\ &= \int_r^t d\theta \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \int_0^\alpha \frac{1}{\sqrt{\alpha-\omega}} \frac{1}{\sqrt{\omega}} d\omega, \\ &\leq 4 \int_r^t \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta, \end{aligned} \quad (3.34)$$

where the latter line above comes from the fact for all $\theta > 0$,

$$\int_0^\theta \frac{1}{\sqrt{\theta-\omega}} \frac{1}{\sqrt{\omega}} d\omega = \int_0^1 \frac{1}{\sqrt{1-\omega}} \frac{1}{\sqrt{\omega}} d\omega = B\left(\frac{1}{2}, \frac{1}{2}\right) = \Gamma^2\left(\frac{1}{2}\right) = \pi,$$

Γ, B denoting respectively the Euler gamma and Beta functions.

Injecting inequality (3.34) in (3.33), we obtain for all $t \in [r, r + \tau]$

$$\|\Pi^2(v_1)(t, \cdot) - \Pi^2(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq C(4 + \tau) \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.35)$$

Iterating previous inequality, one obtains the following. For all $k \geq 1, t \in [r, r + \tau]$,

$$\|\Pi^{2k}(v_1)(t, \cdot) - \Pi^{2k}(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq C^k (4 + \tau)^k \int_r^t \frac{(t-s)^{k-1}}{(k-1)!} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds \quad (3.36)$$

By induction on $k \geq 1$ (3.36) can indeed be established. Finally, by integrating each sides of (3.36) w.r.t. dt and using Fubini's theorem, for $k \geq 1$, we obtain

$$\int_r^{r+\tau} \|\Pi^{2k}(v_1)(t, \cdot) - \Pi^{2k}(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \leq C^k (4 + \tau)^k \frac{T^k}{k!} \int_r^{r+\tau} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.37)$$

For $k_0 \in \mathbb{N}$ large enough, $\frac{(C(4+\tau)T)^{k_0}}{T^{(k_0)!}}$ will be strictly smaller than 1 and Π^{2k_0} will admit a unique fixed-point by Banach fixed-point theorem. In consequence, it implies easily that Π will also admit a unique fixed-point and this concludes the proof of Lemma 3.7. \square

Proof of Theorem 3.6. Without restriction of generality, we can suppose there exists $N \in \mathbb{N}^*$ such that $T = N\tau$, where we recall that τ is given by (3.28). Similarly to the notations used in the preceding proof, in all

the sequel, we agree that for $M > 0$, $B(0, M)$ (resp. $B_\infty(0, M)$) denotes the centered ball of radius M in $W^{1,1}(\mathbb{R}^d)$ (resp. in $L^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$ or in $L^\infty([r, r + \tau] \times \mathbb{R}^d, \mathbb{R})$ for $r \in [0, T - \tau]$ according to the context). The notations introduced in (3.21) will also be used in the present proof.

Indeed, for $r = 0$, $\phi = u_0$ and $M \geq \max(\|u_0\|_\infty; \|u_0\|_1)$, Lemma 3.7 implies there exists a unique function $v^0 : [0, \tau] \times \mathbb{R}^d \rightarrow \mathbb{R}$ (belonging to $L^1([0, \tau], B(0, M)) \cap B_\infty(0, M)$) such that for $(t, x) \in [0, \tau] \times \mathbb{R}^d$,

$$v^0(t, x) = \int_0^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)(v^0(s, x_0) + \widehat{u}_0^0(s, x_0))\Lambda(s, x_0, v^0 + \widehat{u}_0^0, \nabla(v^0 + \widehat{u}_0^0))dx_0, \quad (3.38)$$

where $\widehat{u}_0^0(t, x)$ is given by (3.19) with $\phi = u_0$, i.e.

$$\widehat{u}_0^0(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x)u_0(x_0)dx_0, \quad (t, x) \in [0, \tau] \times \mathbb{R}^d. \quad (3.39)$$

Setting $u^0 := \widehat{u}_0^0 + v^0$, i.e.

$$u^0(t, \cdot) = \int_{\mathbb{R}^d} p(0, x_0, t, \cdot)u_0(x_0)dx_0 + v^0(t, \cdot), \quad t \in [0, \tau], \quad (3.40)$$

it appears that u^0 satisfies for all $(t, x) \in [0, \tau] \times \mathbb{R}^d$

$$u^0(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x)u_0(x_0)dx_0 + \int_0^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)u^0(s, x_0)\Lambda(s, x_0, u^0, \nabla u^0)dx_0. \quad (3.41)$$

Let us fix $k \in \{1, \dots, N - 1\}$. Suppose now given a family of functions u^1, u^2, \dots, u^{k-1} , where for all $j \in \{1, \dots, k - 1\}$, $u^j \in L^1([j\tau, (j + 1)\tau], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([j\tau, (j + 1)\tau] \times \mathbb{R}^d, \mathbb{R})$ and satisfies for all $(t, x) \in [j\tau, (j + 1)\tau] \times \mathbb{R}^d$,

$$u^j(t, x) = \int_{\mathbb{R}^d} p(j\tau, x_0, t, x)u^{j-1}(j\tau, x_0)dx_0 + \int_{j\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)u^j(s, x_0)\Lambda(s, x_0, u^j, \nabla u^j)dx_0. \quad (3.42)$$

Let us introduce

$$\begin{aligned} \widehat{u}_0^k(t, x) &:= \widehat{u}_0(u^{k-1})(t, x) \\ &= \int_{\mathbb{R}^d} p(k\tau, x_0, t, x)u^{k-1}(k\tau, x_0)dx_0, \quad (t, x) \in [k\tau, (k + 1)\tau] \times \mathbb{R}^d, \end{aligned} \quad (3.43)$$

where the second inequality comes from (3.19) with $r = k\tau$ and $\phi = u^{k-1}(k\tau, \cdot)$.

By choosing the real M large enough (i.e. $M \geq \max(\|u^{k-1}(k\tau, \cdot)\|_\infty; \|u^{k-1}(k\tau, \cdot)\|_1)$), Lemma 3.7 applied with $r = k\tau$, $\phi = u^{k-1}(k\tau, \cdot)$ implies existence and uniqueness of a function $v^k : [k\tau, (k + 1)\tau] \times \mathbb{R}^d \rightarrow \mathbb{R}$ that belongs to $L^1([k\tau, (k + 1)\tau], B(0, M)) \cap B_\infty(0, M)$ and satisfying

$$v^k(t, x) = \int_{k\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)(v^k(s, x_0) + \widehat{u}_0^k(s, x_0))\Lambda(s, x_0, v^k + \widehat{u}_0^k, \nabla(v^k + \widehat{u}_0^k))dx_0, \quad (3.44)$$

for all $(t, x) \in [k\tau, (k + 1)\tau] \times \mathbb{R}^d$. Setting $u^k := \widehat{u}_0^k + v^k$, we have for all $(t, x) \in [k\tau, (k + 1)\tau] \times \mathbb{R}^d$

$$u^k(t, x) = \int_{\mathbb{R}^d} p(k\tau, x_0, t, x)u^{k-1}(k\tau, x_0)dx_0 + \int_{k\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x)u^k(s, x_0)\Lambda(s, x_0, u^k, \nabla u^k)dx_0. \quad (3.45)$$

Consequently, by induction we can construct a family of functions $(u^k : [k\tau, (k + 1)\tau] \times \mathbb{R}^d \rightarrow \mathbb{R})_{k=0, \dots, N-1}$ such that for all $k \in \{0, \dots, N - 1\}$, $u^k \in L^1([k\tau, (k + 1)\tau], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([k\tau, (k + 1)\tau] \times \mathbb{R}^d, \mathbb{R})$ and verifies (3.45).

We now consider the real-valued function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined as being equal to u^k (defined by (3.45)) on each interval $[k\tau, (k+1)\tau]$. Then, Lemma 3.4 applied with τ given by (3.28) and

$$\delta = (\alpha_0 := 0 < \dots < \alpha_k := k\tau < \dots < \alpha_N := T = N\tau) \quad , \quad \mu(t, dx) = u(t, x)dx, \quad (3.46)$$

shows that u is a mild solution of (1.1) on $[0, T] \times \mathbb{R}^d$, in the sense of Definition 2.1, item 2. It now remains to ensure that u is indeed the unique mild solution of (1.1) on $[0, T] \times \mathbb{R}^d$ belonging to $L^1([0, T], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$. This follows, in a classical way, by boundedness and Lipschitz property of Λ .

Indeed, if U, V are two mild solutions of (1.1), then very similar computations as the ones done in (3.29), (3.31), and (3.36) to obtain (3.37) give the following. There exists $C := C(\Phi, g, \Lambda, U, V) > 0$ such that

$$\int_0^T \|U(t, \cdot) - V(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \leq (5C)^j \frac{T^{j-1}}{(j-1)!} \int_0^T \|U(s, \cdot) - V(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.47)$$

If we choose $j \in \mathbb{N}^*$ large enough so that $(5C)^j \frac{T^{j-1}}{(j-1)!} < 1$, we obtain $U(t, x) = V(t, x)$ for almost all $(t, x) \in [0, T] \times \mathbb{R}^d$. This concludes the proof of Theorem 3.6. \square

Corollary 3.8. *Under Assumption 2, there exists a unique function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the Feynman-Kac equation (3.15). In particular, such u coincides with the mild solution of (1.1).*

In the case where the function Λ does not depend on ∇u , existence and uniqueness of a solution of (1.1) in the mild sense can be proved under weaker assumptions. This is the object of the following result.

Theorem 3.9. *Assume that Assumption 1 is satisfied. Let $\mathbf{u}_0 \in \mathcal{P}(\mathbb{R}^d)$ admitting a bounded density (still denoted by the same letter). Let Y the the strong solution of (2.10) with prescribed Y_0 .*

We suppose that the transition probability function P (see (2.18)) admits a density p such that $P(s, x_0, t, dx) = p(s, x_0, t, x)dx$, for all $s, t \in [0, T]$, $x_0 \in \mathbb{R}^d$. Λ is supposed to satisfy items 4. and 5. of Assumption 2. Then, there exists a unique mild solution u of (1.1) in $L^1([0, T], L^1(\mathbb{R}^d))$, i.e. u satisfies

$$u(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x)u_0(x_0)dx_0 + \int_0^t \int_{\mathbb{R}^d} p(s, x_0, t, x)u(s, x_0)\Lambda(s, x_0, u(s, x_0))dx_0 ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (3.48)$$

Proof. Since this theorem can be proved in a very similar way as Theorem 3.6 but with simpler computations, we omit the details. \square

4 Existence/uniqueness of the Regularized Feynman-Kac equation

In this section, we introduce a regularized version of PDE (1.1) to which we associate a regularized Feynman-Kac equation corresponding to a regularized version of (3.15). This regularization procedure constitutes a preliminary step for the construction of a particle scheme approximating (3.15). Indeed, as detailed in the next section devoted to the particle approximation, the point dependence of Λ on u and ∇u will require to derive from a discrete measure (based on the particle system) estimates of densities u and their derivatives ∇u , which can classically be achieved by kernel convolution.

Assumption 1 is in force. Let \mathbf{u}_0 be a Borel probability measure on \mathbb{R}^d and Y_0 a random variable distributed according to \mathbf{u}_0 . We consider Y the strong solution of the SDE (2.10).

Let us consider $(K_\varepsilon)_{\varepsilon>0}$, a sequence of mollifiers verifying (2.4) such that K verifies (2.5). Let $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be bounded, Borel measurable. As announced, we introduce the following integro-PDE corresponding to a regularized version of (1.1)

$$\begin{cases} \partial_t \gamma_t = L_t^* \gamma_t + \gamma_t \Lambda(t, x, K_\varepsilon * \gamma_t, \nabla K_\varepsilon * \gamma_t) \\ \gamma_0 = \mathbf{u}_0 . \end{cases} \quad (4.1)$$

The concept of mild solution associated to this type of equation is clarified by the following definition.

Definition 4.1. A Borel measure-valued function $\gamma : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ will be called a **mild solution** of (4.1) if it satisfies, for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \gamma(t, dx) &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) P(0, x_0, t, dx) \\ &+ \int_{[0, t] \times \mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(x) P(s, x_0, t, dx) \right) \Lambda(s, x_0, (K_\varepsilon * \gamma(s, \cdot))(x_0), (\nabla K_\varepsilon * \gamma(s, \cdot))(x_0)) \gamma(s, dx_0) ds . \end{aligned} \quad (4.2)$$

Similarly as Theorem 3.5, we straightforwardly derive the following equivalence result.

Proposition 4.2. Suppose that Assumption 1 is fulfilled. We indicate by Y the unique strong solution of (2.10) with prescribed $Y_0 \sim \mathbf{u}_0$. A Borel measure-valued function $\gamma^\varepsilon : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ is a mild solution of (4.1) if and only if, for all $\varphi \in C_b(\mathbb{R}^d)$, $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \varphi(x) \gamma_t^\varepsilon(dx) = \mathbb{E} \left[\varphi(Y_t) \exp \left(\int_0^t \Lambda(s, Y_s, (K_\varepsilon * \gamma_s^\varepsilon)(Y_s), (\nabla K_\varepsilon * \gamma_s^\varepsilon)(Y_s)) ds \right) \right] . \quad (4.3)$$

Proof. The proof follows the same lines as the proof of Theorem 3.5. First assume that γ^ε satisfies (4.3), we can show that γ^ε is a mild solution (4.1) by imitating the first step of the proof of Proposition 3.3. Secondly, the converse is proved by applying Proposition 3.3 with $\tilde{\Lambda}(t, x) := \Lambda(t, x, (K_\varepsilon * \gamma_t^\varepsilon)(x), (\nabla K_\varepsilon * \gamma_t^\varepsilon)(x))$ and $\mu(t, dx) := \gamma_t^\varepsilon(dx)$. \square

Let us now prove existence and uniqueness of a mild solution for the integro-PDE (4.1). To this end, we proceed similarly as for the proof of Theorem 3.6 using Lemma 3.7. Let $\tau > 0$ be a constant supposed to be fixed for the moment and let us fix $\varepsilon > 0$, $r \in [0, T - \tau]$. $\mathcal{B}([r, r + \tau], \mathcal{M}_f(\mathbb{R}^d))$ denotes the space of bounded, measure-valued maps, where $\mathcal{M}_f(\mathbb{R}^d)$ is equipped with the total variation norm $\|\cdot\|_{TV}$. Given $M > 0$, we denote by $B(0, M)$ denotes the centered ball in $(\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV})$ with radius M and by $\mathcal{B}([r, r + \tau], B(0, M))$, the closed subset of $\mathcal{B}([r, r + \tau], \mathcal{M}_f(\mathbb{R}^d))$ of $B(0, M)$ -valued maps defined on $[r, r + \tau]$. We introduce the measure-valued application $\Pi_\varepsilon : \beta \in \mathcal{B}([r, r + \tau], \mathcal{M}_f(\mathbb{R}^d)) \rightarrow \Pi_\varepsilon(\beta)$, defined by

$$\begin{aligned} \Pi_\varepsilon(\beta)(t, dx) &= \int_r^t \int_{\mathbb{R}^d} P(s, x_0, t, dx) \Lambda(s, x_0, (K_\varepsilon * \hat{\beta}(s, \cdot))(x_0), (\nabla K_\varepsilon * \hat{\beta}(s, \cdot))(x_0)) \hat{\beta}(s, dx_0) ds . \\ \hat{\beta}(s, \cdot) &= \beta(s, \cdot) + \widehat{u}_0(s, \cdot) , \end{aligned} \quad (4.4)$$

where the function \widehat{u}_0 , defined on $[r, r + \tau] \times \mathcal{M}_f(\mathbb{R}^d)$, is given by

$$\widehat{u}_0(r, \pi)(t, dx) := \int_{\mathbb{R}^d} p(r, x_0, t, dx) \pi(dx_0), \quad t \in [r, r + \tau], \quad \pi \in \mathcal{M}_f(\mathbb{R}^d), \quad (4.5)$$

similarly to (3.19). In the sequel, the dependence of \widehat{u}_0 w.r.t. r, π will be omitted when it is self-explanatory.

Lemma 4.3. *Assume the validity of items 4. and 5. of Assumption 2. Let $\pi \in B(0, M)$.*

Let us fix $\varepsilon > 0$ and $M > 0$ such that $M \geq \|\pi\|_{TV}$. Then, there is $\tau > 0$ only depending on M_Λ such that for any $r \in [0, T - \tau]$, Π_ε admits a unique fixed-point in $\mathcal{B}([r, r + \tau], B(0, M))$.

Proof. Let us define $\tau := \frac{1}{2M_\Lambda}$. For every $\lambda \geq 0$, $\mathcal{B}([r, r + \tau], \mathcal{M}_f(\mathbb{R}^d))$ will be equipped with one of the equivalent norms

$$\|\beta\|_{TV, \lambda} := \sup_{t \in [r, r + \tau]} e^{-\lambda t} \|\beta(t, \cdot)\|_{TV}. \quad (4.6)$$

Recalling (4.4), where \widehat{u}_0 is defined by (4.5), it follows that for all $\beta \in \mathcal{B}([r, r + \tau], B(0, M))$, $t \in [r, r + \tau]$,

$$\|\Pi_\varepsilon(\beta)(t, \cdot)\|_{TV} \leq M_\Lambda \int_r^t \|\beta(s, \cdot)\|_{TV} ds + M_\Lambda M \tau \leq 2MM_\Lambda \tau \leq M, \quad (4.7)$$

where for the latter inequality of (4.7) we have used the definition of $\tau := \frac{1}{2M_\Lambda}$. We deduce that $\Pi(\mathcal{B}([r, r + \tau], B(0, M))) \subset \mathcal{B}([r, r + \tau], B(0, M))$.

Consider now $\beta^1, \beta^2 \in \mathcal{B}([r, r + \tau], B(0, M))$. For all $\lambda > 0$ we have

$$\begin{aligned} \|\Pi_\varepsilon(\beta^1(t, \cdot)) - \Pi_\varepsilon(\beta^2(t, \cdot))\|_{TV} &\leq M_\Lambda \int_r^t \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} ds \\ &\quad + L_\Lambda (\|K_\varepsilon\|_\infty + \|\nabla K_\varepsilon\|_\infty) \int_r^t \|\beta^1(s, \cdot)\|_{TV} \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} ds \\ &\quad + L_\Lambda (\|K_\varepsilon\|_\infty + \|\nabla K_\varepsilon\|_\infty) \int_r^t \|\widehat{u}_0(s, \cdot)\|_{TV} \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} ds \\ &\leq C_{\varepsilon, T} \int_0^t \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} ds \\ &\leq C_{\varepsilon, T} \int_0^t e^{s\lambda} \|\beta^1 - \beta^2\|_{TV, \lambda} ds \\ &= C_{\varepsilon, T} \|\beta^1 - \beta^2\|_{TV, \lambda} \frac{e^{\lambda t} - 1}{\lambda}, \end{aligned} \quad (4.8)$$

with $C_{\varepsilon, T} := 2L_\Lambda M (\|K_\varepsilon\|_\infty + \|\nabla K_\varepsilon\|_\infty) + M_\Lambda$. It follows

$$\begin{aligned} \|\Pi_\varepsilon(\beta^1) - \Pi_\varepsilon(\beta^2)\|_{TV, \lambda} &= \sup_{t \in [r, r + \tau]} e^{-\lambda t} \|\Pi(\beta^1)(t, \cdot) - \Pi(\beta^2)(t, \cdot)\|_{TV} \\ &\leq C_{\varepsilon, T} \|\beta^1 - \beta^2\|_{TV, \lambda} \sup_{t \geq 0} \left(\frac{1 - e^{-\lambda t}}{\lambda} \right) \\ &\leq \frac{C_{\varepsilon, T}}{\lambda} \|\beta^1 - \beta^2\|_{TV, \lambda}. \end{aligned} \quad (4.9)$$

Hence, taking $\lambda > C_{\varepsilon, T}$, Π_ε is a contraction on $\mathcal{B}([r, r + \tau], B(0, M))$.

Since $\mathcal{B}([r, r + \tau], (\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV, \lambda}))$ is a Banach space whose $\mathcal{B}([r, r + \tau], B(0, M))$ is a closed subset, the proof ends by a simple application of Banach fixed-point theorem. \square

The next step is to show how the proposition above, with the help of Lemma 3.4, permits us to construct a mild solution of (4.1). The reasoning is similar to the one explained in the proof of Theorem 3.6. Indeed, without restriction of generality, we can suppose there exists $N \in \mathbb{N}^*$ such that $T = N\tau$. Then, for all $k = 0, \dots, N - 1$, Lemma 4.3 applied on each interval $[k\tau, (k + 1)\tau]$ (with $r = k\tau$, $\pi = \beta_\varepsilon^{k-1}(k\tau, \cdot)$ for $k \geq 1$ and $\pi = u_0$ for $k = 0$) gives existence of a family of measure-valued maps $(\beta_\varepsilon^k : [k\tau, (k + 1)\tau] \rightarrow$

$\mathcal{M}_f(\mathbb{R}^d)$, $k = 0, \dots, N-1$) defined by

$$\begin{aligned}\beta_\varepsilon^k(t, dx) &= \int_{k\tau}^t \int_{\mathbb{R}^d} P(s, x_0, t, dx) \Lambda(s, x_0, (K_\varepsilon * \widehat{\beta}_\varepsilon^k(s, \cdot))(x_0), (\nabla K_\varepsilon * \widehat{\beta}_\varepsilon^k(s, \cdot))(x_0)) \widehat{\beta}_\varepsilon^k(s, dx_0) ds, \\ \widehat{\beta}_\varepsilon^k(s, \cdot) &= \beta_\varepsilon^k(s, \cdot) + \widehat{u}_0^k(s, \cdot),\end{aligned}\quad (4.10)$$

where for $k = 0$, $t \in [0, \tau]$,

$$\widehat{u}_0^0(t, dx) = \int_{\mathbb{R}^d} P(0, x_0, t, dx) \mathbf{u}_0(dx_0), \quad \text{by (4.5) with } \pi = u_0, \quad (4.11)$$

and for all $k \in \{1, \dots, N\}$, $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned}\widehat{u}_0^k(t, dx) &:= \widehat{u}_0(\beta_\varepsilon^{k-1})(t, dx) \\ &= \int_{\mathbb{R}^d} P(k\tau, x_0, t, dx) \beta_\varepsilon^{k-1}(k\tau, dx_0), \quad \text{by (4.5) with } \pi = \beta_\varepsilon^{k-1}(k\tau, \cdot).\end{aligned}\quad (4.12)$$

We now consider the following measure-valued maps $\widehat{U}_0 : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ and $\beta_\varepsilon : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ defined by their restrictions on each interval $[k\tau, (k+1)\tau]$, $k = 0, \dots, N-1$ such that

$$\widehat{U}_0(t, x) := \widehat{u}_0^k(t, x) \quad \text{and} \quad \beta_\varepsilon(t, x) := \beta_\varepsilon^k(t, x) \quad \text{for } (t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d, \quad (4.13)$$

and we finally define $\gamma^\varepsilon : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ by

$$\gamma^\varepsilon := \widehat{U}_0 + \beta_\varepsilon \quad \text{on } [0, T] \times \mathbb{R}^d. \quad (4.14)$$

To ensure that γ^ε is indeed a mild solution on $[0, T] \times \mathbb{R}^d$ (in the sense of Definition 4.1) of the integro-PDE (4.1), it is enough to apply Lemma 3.4 with $\tau := \frac{1}{2M_\Lambda}$, $\mu(t, dx) := \gamma^\varepsilon(t, dx)$ and $(\alpha_k := k\tau)_{k=0, \dots, N}$.

Previous discussion leads us to the following proposition.

Proposition 4.4. *Suppose the validity of Assumption 1 and items 4. and 5. of Assumption 2. Let us fix $\varepsilon > 0$ and let γ^ε denote the map defined by (4.14). The following statements hold.*

1. γ^ε is the unique mild solution of the integro-PDE (4.1), see Definition 4.1.
2. γ^ε is the unique solution to the regularized Feynman-Kac equation (4.3).

Proof. The existence of a mild solution γ^ε of (4.1) has already been proved through the discussion just above. It remains to justify uniqueness. Consider $\gamma^{\varepsilon,1}, \gamma^{\varepsilon,2}$ be two mild solutions of (4.2). Then, with similar computations as the ones leading to inequality (4.9), there exists a constant $\mathfrak{C} := \mathfrak{C}(M_\Lambda, L_\Lambda, \|K_\varepsilon\|_\infty, \|\nabla K_\varepsilon\|_\infty) > 0$ such that

$$\|\gamma^{\varepsilon,1} - \gamma^{\varepsilon,2}\|_{TV,\lambda} \leq \frac{\mathfrak{C}}{\lambda} \|\gamma^{\varepsilon,1} - \gamma^{\varepsilon,2}\|_{TV,\lambda}, \quad (4.15)$$

for all $\lambda > 0$ and where we recall that $\|\cdot\|_{TV,\lambda}$ has been defined by (4.6). Taking $\lambda > \mathfrak{C}$, uniqueness follows. This shows item 1. Item 2. follows then by Proposition 4.2. \square

The theorem below states the convergence of the solution of the regularized Feynman-Kac equation (4.3) to the solution to the Feynman-Kac equation (3.15). This is equivalent to the convergence of the solution of the regularized PDE (4.1) to solution of the target PDE (1.1), when the regularization parameter ε goes to zero.

Theorem 4.5. *Suppose the validity of Assumption 2. For any $\varepsilon > 0$, consider the real valued function u^ε such that for any $t \in [0, T]$,*

$$u^\varepsilon(t, \cdot) := K_\varepsilon * \gamma_t^\varepsilon, \quad (4.16)$$

where γ^ε is the unique solution of (4.3) (or equivalently the unique mild solution of (4.1)). Then u^ε converges to u , the unique solution of (3.15) (or equivalently the unique mild solution of (1.1)). More precisely we have

$$\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \text{for any } t \in [0, T]. \quad (4.17)$$

Before proving Theorem 4.5, we state and prove a preliminary lemma.

Lemma 4.6. *Suppose the validity of Assumption 2. Consider u the unique solution of (3.15), then for all $t \in [0, T]$*

$$u(t, x) = F_0(t, x) + \int_0^t \mathbb{E} \left[p(s, Y_s, t, x) \Lambda(s, Y_s, u, \nabla u) e^{\int_0^s \Lambda(r, Y_r, u, \nabla u) dr} \right] ds, \quad dx \text{ a.e.} \quad (4.18)$$

For a given $\varepsilon > 0$, consider u^ε defined by (4.16). Then for almost all $x \in \mathbb{R}^d$ and all $t \in [0, T]$,

$$u^\varepsilon(t, x) = (K_\varepsilon * F_0(t, \cdot))(x) + \int_0^t \mathbb{E} \left[(K_\varepsilon * p(s, Y_s, t, \cdot))(x) \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} \right] ds, \quad (4.19)$$

where $F_0(t, x) := \int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0$ for $t > 0, x \in \mathbb{R}^d$ and $F_0(0, \cdot) := u_0$. We remark that we have used again the notation

$$\Lambda(s, \cdot, v, \nabla v) := \Lambda(s, \cdot, v(s, \cdot), \nabla v(s, \cdot)), \quad t \in [0, T], \quad (4.20)$$

for $v \in L^1([0, T], W^{1,1}(\mathbb{R}^d))$.

Proof. Equalities (4.18) and (4.19) are proved in a very similar way, so we only provide the proof of equation (4.19).

We observe that for all $t \in [0, T], v \in L^1([0, T], W^{1,1}(\mathbb{R}^d))$,

$$e^{\int_0^t \Lambda(s, Y_s, v(s, Y_s), \nabla v(s, Y_s)) ds} = 1 + \int_0^t \Lambda(r, Y_r, v(r, Y_r), \nabla v(r, Y_r)) e^{\int_0^r \Lambda(s, Y_s, v(s, Y_s), \nabla v(s, Y_s)) ds} dr. \quad (4.21)$$

Taking into account the notation introduced in (4.3), (4.16) and (4.20), the identity (4.21) above implies for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned} u^\varepsilon(t, x) &= \mathbb{E} \left[K_\varepsilon(x - Y_t) \exp \left\{ \int_0^t \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds \right\} \right] \\ &= \mathbb{E} \left[K_\varepsilon(x - Y_t) \right] + \int_0^t \mathbb{E} \left[K_\varepsilon(x - Y_t) \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr \\ &= \int_{\mathbb{R}^d} K_\varepsilon(x - y) \int_{\mathbb{R}^d} p(0, x_0, t, y) u_0(x_0) dx_0 dy + \\ &\quad \int_0^t \mathbb{E} \left[\mathbb{E} \left[K_\varepsilon(x - Y_t) \middle| Y_r \right] \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr \\ &= (K_\varepsilon * F_0)(t, \cdot)(x) + \\ &\quad \int_0^t \mathbb{E} \left[\left(\int_{\mathbb{R}^d} K_\varepsilon(x - y) p(r, Y_r, t, y) dy \right) \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr \\ &= (K_\varepsilon * F_0)(t, \cdot)(x) + \\ &\quad \int_0^t \mathbb{E} \left[(K_\varepsilon * p(r, Y_r, t, \cdot))(x) \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr. \end{aligned} \quad (4.22)$$

This ends the proof. \square

Proof of Theorem 4.5. In this proof, C denotes a real constant that may change from line to line, only depending on $M_\Lambda, L_\Lambda, C_u$ and $\|u_0\|_\infty$, where we recall that the constant C_u only depends on Φ, g and come from inequality (6.15).

We first observe that for $t = 0$, the convergence of $u^\varepsilon(0, \cdot)$ (resp. $\nabla u^\varepsilon(0, \cdot)$) to $u(0, \cdot)$ (resp. $\nabla u(0, \cdot)$) in $L^1(\mathbb{R}^d)$ -norm when ε goes to 0 is clear. Let us fix $t \in (0, T]$.

By Lemma 4.6, for almost all $x \in \mathbb{R}^d$, we have the decomposition

$$\begin{aligned} u^\varepsilon(t, x) - u(t, x) &= (K_\varepsilon * F_0(t, \cdot))(x) - F_0(t, x) + \\ &\int_0^t \mathbb{E} \left[\left\{ (K_\varepsilon * p(s, Y_s, t, \cdot))(x) - p(s, Y_s, t, x) \right\} \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} \right] ds + \\ &\int_0^t \mathbb{E} \left[p(s, Y_s, t, x) \left\{ \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} - \Lambda(s, Y_s, u, \nabla u) e^{\int_0^s \Lambda(r, Y_r, u, \nabla u) dr} \right\} \right] ds. \end{aligned} \quad (4.23)$$

By integrating the absolute value of both sides of (4.23) w.r.t. dx , it follows there exists a constant $C > 0$ such that

$$\begin{aligned} \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * F_0 - F_0\|_1 + \int_0^t \mathbb{E} \left[\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds \right. \\ &\quad + \int_0^t \mathbb{E} \left[|u^\varepsilon(s, Y_s) - u(s, Y_s)| + |\nabla u^\varepsilon(s, Y_s) - \nabla u(s, Y_s)| \right] ds \\ &\quad \left. + \int_0^t \int_0^s \mathbb{E} \left[|u^\varepsilon(r, Y_r) - u(r, Y_r)| + |\nabla u^\varepsilon(r, Y_r) - \nabla u(r, Y_r)| \right] dr ds \right\}. \end{aligned} \quad (4.24)$$

Moreover, by (2.11),

$$p_s(x) = \int_{\mathbb{R}^d} p(0, x_0, s, x) u_0(x_0) dx_0, \quad (4.25)$$

is the law density of Y_s , by inequality (6.16) of Lemma 6.4 we get

$$\begin{aligned} \mathbb{E} \left[|u^\varepsilon(s, Y_s) - u(s, Y_s)| \right] &= \int_{\mathbb{R}^d} |u^\varepsilon(s, x) - u(s, x)| p_s(x) dx \\ &\leq C_u \|u_0\|_\infty \int_{\mathbb{R}^d} |u^\varepsilon(s, x) - u(s, x)| dx \\ &= C_u \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1, \quad s \in [0, T], \end{aligned} \quad (4.26)$$

and

$$\begin{aligned} \mathbb{E} \left[|\nabla u^\varepsilon(s, Y_s) - \nabla u(s, Y_s)| \right] &= \int_{\mathbb{R}^d} |\nabla u^\varepsilon(s, x) - \nabla u(s, x)| p_s(x) dx \\ &\leq C_u \|u_0\|_\infty \int_{\mathbb{R}^d} |\nabla u^\varepsilon(s, x) - \nabla u(s, x)| dx \\ &= C_u \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1, \quad s \in [0, T]. \end{aligned} \quad (4.27)$$

Injecting (4.26) and (4.27) into the r.h.s. of (4.24), it comes

$$\begin{aligned} \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * F_0 - F_0\|_1 + \int_0^t \mathbb{E} \left[\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds \right. \\ &\quad \left. + \int_0^t \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1 + \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1 ds \right\}. \end{aligned} \quad (4.28)$$

Now, we need to bound $\|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1$. To this end, we can remark that for almost all $x \in \mathbb{R}^d$,

$$\nabla u(t, x) = \nabla F_0(t, x) + \int_0^t \mathbb{E} \left[\nabla_x p(s, Y_s, t, x) \Lambda(s, Y_s, u, \nabla u) e^{\int_0^s \Lambda(r, Y_r, u, \nabla u) dr} \right] ds, \quad (4.29)$$

and

$$\begin{aligned} \nabla u^\varepsilon(t, x) &= (K_\varepsilon * \nabla F_0(t, \cdot))(x) \\ &\quad + \int_0^t \mathbb{E} \left[(K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot))(x) \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} \right] ds. \end{aligned} \quad (4.30)$$

These equalities follow by computing the derivative of $u(t, \cdot)$ and $u^\varepsilon(t, \cdot)$ in the sense of distributions.

Taking into account (4.29) and (4.30), it is easy to see that very similar arguments as those invoked above to prove (4.28), lead to

$$\begin{aligned} \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 + \int_0^t \mathbb{E} \left[\|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \right] ds \right. \\ &\quad \left. + \int_0^t \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1 + \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1 ds \right\}. \end{aligned} \quad (4.31)$$

Gathering (4.28) together with (4.31) yields

$$\begin{aligned} \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 + \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 + \right. \\ &\quad \int_0^t \mathbb{E} \left[\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds + \\ &\quad \left. \int_0^t \mathbb{E} \left[\|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \right] ds \right\} \\ &\quad + \int_0^t \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1 + \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1 ds. \end{aligned} \quad (4.32)$$

Applying Gronwall's lemma to the real-valued function

$$t \mapsto \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1,$$

we obtain

$$\begin{aligned} \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 &\leq C e^{CT} \left\{ \|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 + \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 + \right. \\ &\quad \int_0^t \mathbb{E} \left[\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds + \\ &\quad \left. \int_0^t \mathbb{E} \left[\|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \right] ds \right\}. \end{aligned} \quad (4.33)$$

Since $F_0(t, \cdot)$, $\nabla F_0(t, \cdot)$, $x \mapsto p(s, x_0, t, x)$ and $x \mapsto \nabla_x p(s, x_0, t, x)$ belong to $L^1(\mathbb{R}^d)$, classical properties of convergence of the mollifiers give

$$\|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 \rightarrow 0, \quad \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 \rightarrow 0, \quad (4.34)$$

and

$$\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \rightarrow 0, \quad \|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \rightarrow 0, \quad a.s. \quad (4.35)$$

Moreover, by inequalities (6.14) and (6.15) of Lemma 6.4, for $0 \leq s < t \leq T$,

$$\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 + \|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \leq 2C_u \left(1 + \frac{1}{\sqrt{t-s}} \right) a.s. \quad (4.36)$$

Lebesgue dominated convergence theorem then implies that the third and fourth terms in the r.h.s. of (4.33) converge to 0 when ε goes to 0. This ends the proof. \square

Proposition 4.7. *We assume here that K verifies (2.6). Let u^ε be the real-valued function defined by (4.16). Under Assumption 2 and in the particular case where the function $(t, x, y, z) \mapsto \Lambda(t, x, y, z)$ does not depend on the z variable corresponding to the gradient ∇u , there exists a constant*

$$C := C(M_\Lambda, L_\Lambda, C_u, \|u_0\|_\infty, \kappa) > 0, \quad (4.37)$$

with C_u denoting the constant given by (6.14) (only depending on Φ, g) such that the following holds. For all $t \in]0, T]$,

$$\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 \leq \varepsilon C \left(\frac{1}{\sqrt{t}} + 2\sqrt{t} \right), \quad (4.38)$$

Proof. In the proof C is a constant fulfilling (4.37). The arguments are the same as the ones used in the proof of Theorem 4.5 since in the present case, Λ only depends on (t, x, u) and not on ∇u . In particular, we obtain for $t \in]0, T]$,

$$\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 \leq C e^{CT} \|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 + \int_0^t \mathbb{E} \left[\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds, \quad (4.39)$$

that corresponds to inequality (4.33) in the proof above, without the term containing the gradient ∇u . Invoking inequality (6.14) of Lemma 6.4 and inequality (6.8) of Lemma 6.3 with $H = K$, and successively with $f = F_0(t, \cdot)$ and $f = p(s, y, t, \cdot)$ for fixed $y \in \mathbb{R}^d$, imply that

$$\|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 \leq \frac{C\varepsilon}{\sqrt{t}}, \quad 0 < t \leq T, \quad (4.40)$$

and

$$\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \leq \frac{C\varepsilon}{\sqrt{t-s}}, \text{ a.e. } 0 \leq s < t \leq T. \quad (4.41)$$

This concludes the proof of (4.38). \square

5 Particles system algorithm

To simplify notations in the rest of the paper, f_t will denote $f(t)$ where $f : [0, T] \rightarrow E$ is an E -valued Borel function and (E, d_E) a metric space.

In previous sections, we have studied existence, uniqueness for a semilinear PDE of the form (1.1) and we have established a Feynman-Kac type representation for the corresponding solution u , see Theorem 3.5. The regularized form of (1.1) is the integro-PDE (4.1) for which we have established well-posedness in Proposition 4.4 1. In the sequel, we denote by γ^ε the corresponding solution and again by $u^\varepsilon(t, x) := (K_\varepsilon * \gamma_t^\varepsilon)(x)$, see (4.16). We recall that u^ε converges to u , the unique solution of (3.15) (or equivalently the unique mild solution of (1.1)), when the regularization parameter ε vanishes to 0, see Theorem 4.5. In the present section, we propose a Monte Carlo approximation $u^{\varepsilon, N}$ of u^ε , providing an original numerical approximation of the semilinear PDE (1.1), when the regularization parameter $\varepsilon \rightarrow 0$ slowly enough, while the number of particles $N \rightarrow \infty$. Let \mathbf{u}_0 be a Borel probability measure on $\mathcal{P}(\mathbb{R}^d)$.

5.1 Convergence of the particle system

We suppose the validity of Assumption 2.

For fixed $N \in \mathbb{N}^*$, let $(W^i)_{i=1, \dots, N}$ be a family of independent Brownian motions and $(Y_0^i)_{i=1, \dots, N}$ be i.i.d. random variables distributed according to $u_0(x)dx$. For any $\varepsilon > 0$, we define the measure-valued

functions $(\gamma_t^{\varepsilon,N})_{t \in [0,T]}$ such that for any $t \in [0, T]$

$$\begin{cases} \xi_t^i = \xi_0^i + \int_0^t \Phi(s, \xi_s^i) dW_s^i + \int_0^t g(s, \xi_s^i) ds, & \text{for } i = 1, \dots, N, \\ \xi_0^i = Y_0^i & \text{for } i = 1, \dots, N, \\ \gamma_t^{\varepsilon,N} = \frac{1}{N} \sum_{i=1}^N V_t(\xi^i, (K_\varepsilon * \gamma^{\varepsilon,N})(\xi^i), (\nabla K_\varepsilon * \gamma^{\varepsilon,N})(\xi^i)) \delta_{\xi_t^i}, \end{cases} \quad (5.1)$$

where $(K_\varepsilon)_{\varepsilon > 0}$ are mollifiers fulfilling (2.4) and (2.5). We recall that V_t is given by (2.1). The first line of (5.1) is a d -dimensional classical SDE whose strong existence and pathwise uniqueness are ensured by classical theorems for Lipschitz coefficients. Moreover $\xi^i, i = 1, \dots, N$ are i.i.d. In the following lemma, we prove by a fixed-point argument that the third line equation of (5.1) has a unique solution.

Lemma 5.1. *We suppose the validity of Assumption 2. Let us fix $\varepsilon > 0$ and $N \in \mathbb{N}^*$. Consider the i.i.d. system $(\xi^i)_{i=1, \dots, N}$ of particles, solution of the two first equations of (5.1). Then, there exists a unique function $\gamma^{\varepsilon,N} : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$ such that for all $t \in [0, T]$, $\gamma_t^{\varepsilon,N}$ is solution of (5.1).*

Proof. The proof relies on a fixed-point argument applied to the map $T^{\varepsilon,N} : \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)) \rightarrow \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d))$ given by

$$T^{\varepsilon,N}(\gamma)(t) : \gamma \mapsto \frac{1}{N} \sum_{i=1}^N V_t(\xi^i, (K_\varepsilon * \gamma)(\xi^i), (\nabla K_\varepsilon * \gamma)(\xi^i)) \delta_{\xi_t^i}. \quad (5.2)$$

In the rest of the proof, the notation $T_t^{\varepsilon,N}(\gamma)$ will denote $T^{\varepsilon,N}(\gamma)(t)$.

In order to apply the Banach fixed-point theorem, we emphasize that $\mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d))$ is equipped with one of the equivalent norms $\|\cdot\|_{TV,\lambda}$, $\lambda \geq 0$, defined by

$$\|\gamma\|_{TV,\lambda} := \sup_{t \in [0, T]} e^{-\lambda t} \|\gamma(t, \cdot)\|_{TV}, \quad (5.3)$$

and for which $(\mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)), \|\cdot\|_{TV,\lambda})$ is still complete.

From now on, it remains to ensure that $T^{\varepsilon,N}$ is indeed a contraction with respect $\|\gamma\|_{TV,\lambda}$ for some λ . To simplify notations, we set for all $i \in \{1, \dots, N\}$,

$$T_t^{\varepsilon,N,i}(\gamma) := V_t(\xi^i, (K_\varepsilon * \gamma)(\xi^i), (\nabla K_\varepsilon * \gamma)(\xi^i)), \quad (t, \gamma) \in [0, T] \times \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)), \quad (5.4)$$

to re-write (5.2) in the form

$$T_t^{\varepsilon,N}(\gamma) = \frac{1}{N} \sum_{i=1}^N T_t^{\varepsilon,N,i}(\gamma) \delta_{\xi_t^i}, \quad (t, \gamma) \in [0, T] \times \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)). \quad (5.5)$$

Let $\lambda > 0$. Consider now $\gamma^1, \gamma^2 \in \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d))$. On the one hand, taking into account (2.1) and (2.3), for all $t \in [0, T]$, $i \in \{1, \dots, N\}$, we have

$$\begin{aligned} |T_t^{\varepsilon,N,i}(\gamma^1) - T_t^{\varepsilon,N,i}(\gamma^2)| &\leq L_\Lambda e^{TM_\Lambda} \int_0^t \left(|(K_\varepsilon * \gamma^1)(\xi_s^i) - (K_\varepsilon * \gamma^2)(\xi_s^i)| \right. \\ &\quad \left. + |(\nabla K_\varepsilon * \gamma^1)(\xi_s^i) - (\nabla K_\varepsilon * \gamma^2)(\xi_s^i)| \right) ds \\ &\leq C \int_0^t \|\gamma_s^1 - \gamma_s^2\|_{TV} ds \\ &\leq C \int_0^t e^{s\lambda} \|\gamma^1 - \gamma^2\|_{TV,\lambda} ds \\ &= C \|\gamma^1 - \gamma^2\|_{TV,\lambda} \frac{e^{\lambda t} - 1}{\lambda}, \end{aligned} \quad (5.6)$$

with $C = C(T, \|K_\varepsilon\|_\infty, \|\nabla K_\varepsilon\|_\infty, L_\Lambda, M_\Lambda)$. It follows that

$$\begin{aligned} \|T^{\varepsilon,N}(\gamma^1) - T^{\varepsilon,N}(\gamma^2)\|_{TV,\lambda} &\leq \frac{1}{N} \sum_{i=1}^N \|T^{\varepsilon,N,i}(\gamma^1)\delta_{\xi^i} - T^{\varepsilon,N,i}(\gamma^2)\delta_{\xi^i}\|_{TV,\lambda} \\ &\leq \frac{C}{\lambda} \|\gamma^1 - \gamma^2\|_{TV,\lambda}. \end{aligned} \quad (5.7)$$

By taking $\lambda > C$ and invoking Banach fixed-point theorem, we end the proof. \square

After the preceding preliminary considerations, we can state and prove the main result of the section.

Proposition 5.2. *We suppose the validity of Assumption 2. Assume that the kernel K is verifying (2.7). Let u^ε be the real valued function defined by (4.16), and $u^{\varepsilon,N}$ such that for any $t \in [0, T]$,*

$$u^{\varepsilon,N}(t, \cdot) := K_\varepsilon * \gamma_t^{\varepsilon,N}, \quad (5.8)$$

where $\gamma^{\varepsilon,N}$ is defined by the third line of (5.1). There is a constant C (only depending on $M_\Phi, M_g, M_\Lambda, \|K\|_\infty, \|\nabla K\|_\infty, L_\Phi, L_g, L_\Lambda, T, \|u_0\|_\infty$ and C_u) such that the following holds.

1. For all $t \in [0, T]$ and $N \in \mathbb{N}^*$, $\varepsilon > 0$ verifying $\min(N, N\varepsilon^d) > C$ we have

$$\mathbb{E} \left[\|u_t^{\varepsilon,N} - u_t^\varepsilon\|_1 \right] + \mathbb{E} \left[\|\nabla u_t^{\varepsilon,N} - \nabla u_t^\varepsilon\|_1 \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}}. \quad (5.9)$$

2. In the particular case where the function $(t, x, y, z) \mapsto \Lambda(t, x, y, z)$ does not depend on the z variable (corresponding to the gradient ∇u), then previous item holds replacing (5.9) with

$$\mathbb{E} \left[\|u_t^{\varepsilon,N} - u_t^\varepsilon\|_1 \right] \leq \frac{C}{\sqrt{N\varepsilon^d}}. \quad (5.10)$$

Proof. We begin by establishing the proof of (5.9), in the general case where Λ may depend both on u and ∇u . Let us fix $\varepsilon > 0$, $N \in \mathbb{N}^*$. For any $\ell = 1, \dots, d$, we introduce the real-valued function G_ε^ℓ defined on \mathbb{R}^d such that

$$G_\varepsilon^\ell(x) := \frac{1}{\varepsilon^d} \frac{\partial K}{\partial x_\ell} \left(\frac{x}{\varepsilon} \right), \quad \text{for almost all } x \in \mathbb{R}^d. \quad (5.11)$$

By (2.7), there exists a finite positive constant C independent of ε such that $\|G_\varepsilon^\ell\|_\infty \leq \frac{C}{\varepsilon^d}$ and $\|G_\varepsilon^\ell\|_1 = \|G_1^\ell\|_1 \leq C$. In the sequel, C will always denote a finite positive constant independent of (ε, N) that may change from line to line. For any $t \in [0, T]$, we introduce the random Borel measure $\tilde{\gamma}_t^{\varepsilon,N}$ on \mathbb{R}^d , defined by

$$\tilde{\gamma}_t^{\varepsilon,N} := \frac{1}{N} \sum_{i=1}^N V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \delta_{\xi^i}. \quad (5.12)$$

One can first decompose the error on the l.h.s of inequality (5.9) as follows

$$\begin{aligned} \mathbb{E} \left[\|u_t^{\varepsilon,N} - u_t^\varepsilon\|_1 \right] + \mathbb{E} \left[\|\nabla u_t^{\varepsilon,N} - \nabla u_t^\varepsilon\|_1 \right] &= \mathbb{E} \left[\|K_\varepsilon * (\gamma_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1 \right] + \frac{1}{\varepsilon} \sum_{\ell=1}^d \mathbb{E} \left[\|G_\varepsilon^\ell * (\gamma_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1 \right] \\ &\leq \mathbb{E} \left[\|K_\varepsilon * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})\|_1 \right] + \frac{1}{\varepsilon} \sum_{\ell=1}^d \mathbb{E} \left[\|G_\varepsilon^\ell * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})\|_1 \right] \\ &\quad + \mathbb{E} \left[\|K_\varepsilon * (\tilde{\gamma}_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1 \right] + \frac{1}{\varepsilon} \sum_{\ell=1}^d \mathbb{E} \left[\|G_\varepsilon^\ell * (\tilde{\gamma}_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1 \right] \\ &= \mathbb{E} \left[\|A_t^{\varepsilon,N}\|_1 \right] + \mathbb{E} \left[\|A_t^{\prime\varepsilon,N}\|_1 \right] + \mathbb{E} \left[\|B_t^{\varepsilon,N}\|_1 \right] + \mathbb{E} \left[\|B_t^{\prime\varepsilon,N}\|_1 \right], \end{aligned} \quad (5.13)$$

where, for all $t \in [0, T]$,

$$\left\{ \begin{array}{l} A_t^{\varepsilon, N}(x) := \frac{1}{N} \sum_{i=1}^N K_\varepsilon(x - \xi_t^i) \left[V_t(\xi^i, u^{\varepsilon, N}(\xi^i), \nabla u^{\varepsilon, N}(\xi^i)) - V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \right] \\ A_t^{\prime \varepsilon, N}(x) := \frac{1}{\varepsilon} \sum_{\ell=1}^d \left| \frac{1}{N} \sum_{i=1}^N G_\varepsilon^\ell(x - \xi_t^i) \left[V_t(\xi^i, u^{\varepsilon, N}(\xi^i), \nabla u^{\varepsilon, N}(\xi^i)) - V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \right] \right| \\ B_t^{\varepsilon, N}(x) := \frac{1}{N} \sum_{i=1}^N K_\varepsilon(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E} \left[K_\varepsilon(x - \xi_t^1) V_t(\xi^1, u^\varepsilon(\xi^1), \nabla u^\varepsilon(\xi^1)) \right] \\ B_t^{\prime \varepsilon, N}(x) := \frac{1}{\varepsilon} \sum_{\ell=1}^d \left| \frac{1}{N} \sum_{i=1}^N G_\varepsilon^\ell(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E} \left[G_\varepsilon^\ell(x - \xi_t^1) V_t(\xi^1, u^\varepsilon(\xi^1), \nabla u^\varepsilon(\xi^1)) \right] \right|. \end{array} \right. \quad (5.14)$$

We will proceed in two steps, first bounding $\mathbb{E}[\|B_t^{\varepsilon, N}\|_1]$ and $\mathbb{E}[\|B_t^{\prime \varepsilon, N}\|_1]$ and then $\mathbb{E}[\|A_t^{\varepsilon, N}\|_1]$ and $\mathbb{E}[\|A_t^{\prime \varepsilon, N}\|_1]$.

Step 1. Bounding $\mathbb{E}\|B_t^{\varepsilon, N}\|_1$ and $\mathbb{E}\|B_t^{\prime \varepsilon, N}\|_1$. For any $i \in \{1, \dots, N\}$ and $(t, x) \in [0, T] \times \mathbb{R}^d$ we set

$$P_i^\varepsilon(t, x) := K_\varepsilon(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E} \left[K_\varepsilon(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \right]. \quad (5.15)$$

Notice that for fixed $(t, x) \in [0, T] \times \mathbb{R}^d$, $(P_i^\varepsilon(t, x))_{i=1, \dots, N}$ are i.i.d. centered square integrable random variables. Hence using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}[\|B_t^{\varepsilon, N}\|_1] &= \int_{\mathbb{R}^d} \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N P_i^\varepsilon(t, x) \right\| \right] dx \\ &\leq \int_{\mathbb{R}^d} \sqrt{\mathbb{E} \left[\left(\frac{1}{N} \sum_{i=1}^N (P_i^\varepsilon(t, x)) \right)^2 \right]} dx \\ &= \frac{1}{\sqrt{N}} \int_{\mathbb{R}^d} \sqrt{\mathbb{E} \left[(P_1^\varepsilon(t, x))^2 \right]} dx. \end{aligned} \quad (5.16)$$

By the boundedness assumption on Λ (item 5. of Assumption 2.),

$$\mathbb{E}[(P_1^\varepsilon(t, x))^2] \leq 4e^{2M\Lambda T} \mathbb{E}[(K_\varepsilon(x - \xi_t^1))^2],$$

which implies

$$\mathbb{E}[\|B_t^{\varepsilon, N}\|_1] \leq \frac{C}{\sqrt{N}} \int_{\mathbb{R}^d} \sqrt{\mathbb{E}[(K_\varepsilon(x - \xi_t^1))^2]} dx = \frac{C}{\sqrt{N}} \frac{\sqrt{\int_{\mathbb{R}^d} K^2(x) dx}}{\sqrt{\varepsilon^d}} \int_{\mathbb{R}^d} \sqrt{H_\varepsilon * p_t}(x) dx, \quad (5.17)$$

where we recall that p_t , defined in (4.25), is the law density of Y_t (or ξ_t^1). Moreover H_ε is the probability density on \mathbb{R}^d such that for almost all $x \in \mathbb{R}^d$, $H_\varepsilon(x) := \frac{1}{\int_{\mathbb{R}^d} K^2(x) dx} \frac{1}{\varepsilon^d} K^2\left(\frac{x}{\varepsilon}\right)$, which is well-defined thanks to assumption (2.7). Finally, applying Lemma 6.2 with $G = \frac{K^2}{\|K\|_2^2}$ and $f = p_t$ we obtain

$$\mathbb{E}[\|B_t^{\varepsilon, N}\|_1] \leq \frac{C}{\sqrt{N\varepsilon^d}}, \quad \text{for } \varepsilon \text{ small enough.} \quad (5.18)$$

Proceeding similarly for the term $B_t^{\prime \varepsilon, N}$ leads to

$$\mathbb{E}[\|B_t^{\prime \varepsilon, N}\|_1] \leq \frac{C}{\sqrt{N\varepsilon^2}} \sum_{\ell=1}^d \int_{\mathbb{R}^d} \sqrt{\mathbb{E}[(G_\varepsilon^\ell(x - \xi_t^1))^2]} dx = \frac{C}{\sqrt{N\varepsilon^2}} \sum_{\ell=1}^d \frac{\sqrt{\int_{\mathbb{R}^d} \left| \frac{\partial K}{\partial x_\ell}(x) \right|^2 dx}}{\sqrt{\varepsilon^d}} \int_{\mathbb{R}^d} \sqrt{H_\varepsilon^\ell * p_t}(x) dx, \quad (5.19)$$

where $H_\varepsilon^\ell, \ell = 1, \dots, d$ denotes the probability densities on \mathbb{R}^d such that for almost all $x \in \mathbb{R}^d$, $H_\varepsilon^\ell(x) := \frac{1}{\int_{\mathbb{R}^d} |\frac{\partial K}{\partial x_\ell}(x)|^2 dx} \frac{1}{\varepsilon^d} |\frac{\partial K}{\partial x_\ell}(\frac{x}{\varepsilon})|^2$. Applying again Lemma 6.2 with $G = \frac{|\frac{\partial K}{\partial x_\ell}|^2}{\|\frac{\partial K}{\partial x_\ell}\|_2^2}, \ell = 1, \dots, d$ and f being the $f = p_t$ we obtain

$$\mathbb{E} \left[\|B_t^{\varepsilon, N}\|_1 \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}}, \quad \text{for } \varepsilon \text{ small enough.} \quad (5.20)$$

Step 2. Bounding $\mathbb{E}\|A_t^{\varepsilon, N}\|_1$ and $\mathbb{E}\|A_t^{\prime\varepsilon, N}\|_1$. Recall that $A_t^{\varepsilon, N}(x) = K_\varepsilon * (\gamma_t^{\varepsilon, N} - \tilde{\gamma}_t^{\varepsilon, N})(x)$ and $A_t^{\prime\varepsilon, N}(x) = \frac{1}{\varepsilon} \sum_{\ell=1}^d |G_\varepsilon^\ell * (\gamma_t^{\varepsilon, N} - \tilde{\gamma}_t^{\varepsilon, N})|(x)$, which yields

$$\mathbb{E} \left[\|A_t^{\varepsilon, N}\|_1 \right] + \mathbb{E} \left[\|A_t^{\prime\varepsilon, N}\|_1 \right] \leq \frac{C}{\varepsilon} \mathbb{E} \left[\|\gamma_t^{\varepsilon, N} - \tilde{\gamma}_t^{\varepsilon, N}\|_{TV} \right]. \quad (5.21)$$

We are now interested in bounding the r.h.s. of (5.21).

Recalling (5.1), (5.12) and inequality (2.3), we have

$$\begin{aligned} \mathbb{E} \left[\|\gamma_t^{\varepsilon, N} - \tilde{\gamma}_t^{\varepsilon, N}\|_{TV} \right] &= \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[|V_t(\xi^i, u^{\varepsilon, N}, \nabla u^{\varepsilon, N}) - V_t(\xi^i, u^\varepsilon, \nabla u^\varepsilon)| \right] \\ &\leq C \mathbb{E} \left[\int_0^t (|u_s^{\varepsilon, N} - u_s^\varepsilon|(\xi_s^1) + |\nabla u_s^{\varepsilon, N} - \nabla u_s^\varepsilon|(\xi_s^1)) ds \right] \\ &\leq C \int_0^t \left(\mathbb{E} \left[|K_\varepsilon * (\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N})(\xi_s^1)| \right] + \mathbb{E} \left[|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon, N} - \gamma_s^\varepsilon)(\xi_s^1)| \right] \right) ds, \\ &\quad + \frac{C}{\varepsilon} \sum_{\ell=1}^d \int_0^t \left(\mathbb{E} \left[|G_\varepsilon^\ell * (\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N})(\xi_s^1)| \right] + \mathbb{E} \left[|G_\varepsilon^\ell * (\tilde{\gamma}_s^{\varepsilon, N} - \gamma_s^\varepsilon)(\xi_s^1)| \right] \right) ds. \end{aligned} \quad (5.22)$$

By inequality (6.16) in Lemma 6.4 and (4.25), $\|p_s\|_\infty \leq C_u \|u_0\|_\infty$ for all $s \in [0, T]$. Recalling that γ^ε verifies (4.3), using inequality (5.18), we obtain

$$\begin{aligned} &\mathbb{E} \left[|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon, N} - \gamma_s^\varepsilon)(\xi_s^1)| \right] \\ &\leq \frac{1}{N} \mathbb{E} \left[|K_\varepsilon(0) V_s(\xi^1, u^\varepsilon(\xi^1), \nabla u^\varepsilon(\xi^1)) - \mathbb{E}[K_\varepsilon(\xi_s^1 - \xi_s^2) V_s(\xi^2, u^\varepsilon(\xi^2), \nabla u^\varepsilon(\xi^2)) | \xi^1]| \right] \\ &\quad + \frac{N-1}{N} \frac{1}{N-1} \int_{\mathbb{R}^d} \left| \sum_{i=2}^N \left[K_\varepsilon(x - \xi_s^i) V_s(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E}[K(x - \xi_s^i) V_s(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i))] \right] \right| p_s(x) dx \\ &\leq \frac{C}{N\varepsilon^d} + \frac{N-1}{N} \frac{C}{\sqrt{(N-1)\varepsilon^d}} \\ &\leq \frac{C}{\sqrt{N\varepsilon^d}} \quad (\text{for } (N \text{ and } N\varepsilon^d) \text{ sufficiently large}), s \in [0, T]. \end{aligned} \quad (5.23)$$

Similarly we get

$$\sum_{\ell=1}^d \mathbb{E} \left[\left| \frac{1}{\varepsilon} G_\varepsilon^\ell * (\tilde{\gamma}_s^{\varepsilon, N} - \gamma_s^\varepsilon)(\xi_s^1) \right| \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}}, \quad s \in [0, T]. \quad (5.24)$$

Moreover, for all $s \in [0, T]$, the boundedness of $|K|$ and $|\nabla K|$ implies

$$\mathbb{E} \left[|K_\varepsilon * (\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N})(\xi_s^1)| \right] + \sum_{\ell=1}^d \mathbb{E} \left[\left| \frac{1}{\varepsilon} G_\varepsilon^\ell * (\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N})(\xi_s^1) \right| \right] \leq \frac{C}{\varepsilon^{d+1}} \left[\|\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N}\|_{TV} \right]. \quad (5.25)$$

Injecting inequalities (5.23) (5.24) and (5.25) into (5.22) gives

$$\mathbb{E} \left[\|\gamma_t^{\varepsilon, N} - \tilde{\gamma}_t^{\varepsilon, N}\|_{TV} \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}} + \frac{C}{\varepsilon^{d+1}} \int_0^t \mathbb{E} \left[\|\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N}\|_{TV} \right] ds.$$

By Gronwall's lemma we obtain $\mathbb{E}\left[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^\varepsilon\|_{TV}\right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}} e^{\frac{C}{\varepsilon^{d+1}}}$, which together with (5.21) completes the proof of (5.9) by implying the inequality

$$\mathbb{E}[\|A_t^{\varepsilon,N}\|_1] + \mathbb{E}[\|A_t^{\prime\varepsilon,N}\|_1] \leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}}. \quad (5.26)$$

Now let us treat the proof of (5.10), in the specific case where Λ does not depend on ∇u . Adapting (5.13) when ∇u does not appear in Λ yields

$$\mathbb{E}\left[\|u_t^{\varepsilon,N} - u_t^\varepsilon\|_1\right] \leq \mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|B_t^{\varepsilon,N}\|_1\right], \quad (5.27)$$

where A_t and B_t are given by (5.14). To bound $\mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right]$, we rely again on (5.22), which gives

$$\begin{aligned} \mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right] &= \mathbb{E}\left[\|K_\varepsilon * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})\|_1\right] \\ &\leq \mathbb{E}\left[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N}\|_{TV}\right] \\ &\leq C \int_0^t \mathbb{E}\left[\|(u_s^{\varepsilon,N} - u_s^\varepsilon)(\xi_s^1)\| ds\right] \\ &= C \int_0^t \mathbb{E}\left[\|(K_\varepsilon * \gamma_s^{\varepsilon,N})(\xi_s^1) - u_s^\varepsilon(\xi_s^1)\| ds\right]. \end{aligned} \quad (5.28)$$

Considering an additional particle ξ^0 such that $(\xi^0, \xi^1, \dots, \xi^N)$ are i.i.d. yields

$$\begin{aligned} \mathbb{E}\left[\|(K_\varepsilon * \gamma_s^{\varepsilon,N})(\xi_s^1) - u_s^\varepsilon(\xi_s^1)\|\right] &= \mathbb{E}\left[\left|\frac{1}{N} \sum_{i=1}^N V_s(\xi^i, u^{\varepsilon,N}(\xi^i)) K_\varepsilon(\xi_s^1 - \xi_s^i) - u_s^\varepsilon(\xi_s^1)\right|\right] \\ &\leq \mathbb{E}\left[\left|\frac{1}{N} V_s(\xi^1, u^{\varepsilon,N}(\xi^1)) K_\varepsilon(0)\right|\right] \\ &\quad + \mathbb{E}\left[\left|\frac{1}{N} \sum_{i=2}^N V_s(\xi^i, u^{\varepsilon,N}(\xi^i)) K_\varepsilon(\xi_s^1 - \xi_s^i) - u_s^\varepsilon(\xi_s^1)\right|\right] \\ &\leq \frac{C}{N\varepsilon^d} + \mathbb{E}\left[\left|\frac{1}{N} \sum_{i=2}^N V_s(\xi^i, u^{\varepsilon,N}(\xi^i)) K_\varepsilon(\xi_s^0 - \xi_s^i) - u_s^\varepsilon(\xi_s^0)\right|\right] \\ &\leq \frac{2C}{N\varepsilon^d} + \mathbb{E}\left[\left|\frac{1}{N} \sum_{i=1}^N V_s(\xi^i, u^{\varepsilon,N}(\xi^i)) K_\varepsilon(\xi_s^0 - \xi_s^i) - u_s^\varepsilon(\xi_s^0)\right|\right] \\ &= \frac{2C}{N\varepsilon^d} + C \int_0^t \mathbb{E}\left[\|(K_\varepsilon * \gamma_s^{\varepsilon,N})(\xi_s^0) - (K_\varepsilon * \gamma_s^\varepsilon)(\xi_s^0)\| ds\right]. \end{aligned}$$

Injecting the above inequality in (5.28) and using triangle inequality yields (reminding that C is a constant that may change from line to line)

$$\begin{aligned} \mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right] &\leq \frac{C}{N\varepsilon^d} + C \int_0^t \left(\mathbb{E}\left[\|K_\varepsilon * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})(\xi_s^0)\|_1\right] + \mathbb{E}\left[\|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^0)\|_1\right] \right) ds \\ &\leq \frac{C}{N\varepsilon^d} + C \int_0^t \|p_s\|_\infty \left(\mathbb{E}\left[\|K_\varepsilon * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})\|_1\right] + \mathbb{E}\left[\|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)\|_1\right] \right) ds \\ &= \frac{C}{N\varepsilon^d} + C \int_0^t \|p_s\|_\infty \left(\mathbb{E}\left[\|A_s^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|B_s^{\varepsilon,N}\|_1\right] \right) ds. \end{aligned} \quad (5.29)$$

Using the fact that $\|p_s\|_\infty \leq C_u \|u_0\|_\infty$ by (6.16) and inequality (5.18), implies that for ε small enough we obtain

$$\mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|B_t^{\varepsilon,N}\|_1\right] \leq \frac{C}{\sqrt{N\varepsilon^d}} + C \int_0^t \left(\mathbb{E}\left[\|A_s^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|B_s^{\varepsilon,N}\|_1\right] \right) ds. \quad (5.30)$$

Gronwall's lemma gives

$$\mathbb{E} \left[\|A_t^{\varepsilon, N}\|_1 \right] + \mathbb{E} \left[\|A_t^{\varepsilon, N}\|_1 \right] \leq \frac{C}{\sqrt{N\varepsilon^d}}. \quad (5.31)$$

□

Corollary 5.3. *We suppose the validity of Assumption 2. Let Assume that the kernel K is verifying (2.6) and (2.7).*

1. *If $\varepsilon \rightarrow 0, N \rightarrow +\infty$ such that $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \rightarrow 0$, (where C is the constant coming from Proposition 5.2) then*

$$\mathbb{E} \left[\|u_t^{\varepsilon, N} - u_t\|_1 \right] + \mathbb{E} \left[\|\nabla u_t^{\varepsilon, N} - \nabla u_t\|_1 \right] \rightarrow 0. \quad (5.32)$$

2. *In the particular case where the function $(t, x, y, z) \mapsto \Lambda(t, x, y, z)$ does not depend on the z variable (corresponding to the gradient ∇u), there is a constant C (only depending on $\kappa, C_u, M_\Phi, M_g, M_\Lambda, \|K\|_\infty, \|\nabla K\|_\infty, L_\Phi, L_g, L_\Lambda, T, \|u_0\|_\infty$), such that the following holds.*

$$\mathbb{E} \left[\|u_t^{\varepsilon, N} - u_t\|_1 \right] \leq C \left(\varepsilon + \frac{1}{\sqrt{N\varepsilon^d}} \right). \quad (5.33)$$

Proof. Let us fix $\varepsilon > 0, N \in \mathbb{N}^*, t \in [0, T]$. The proof of (5.32) is based on the bound

$$\begin{aligned} \mathbb{E} \left[\|u_t^{\varepsilon, N} - u_t\|_1 \right] + \mathbb{E} \left[\|\nabla u_t^{\varepsilon, N} - \nabla u_t\|_1 \right] &\leq \mathbb{E} \left[\|u_t^{\varepsilon, N} - u_t^\varepsilon\|_1 \right] + \mathbb{E} \left[\|\nabla u_t^{\varepsilon, N} - \nabla u_t^\varepsilon\|_1 \right] \\ &\quad + \|u_t^\varepsilon - u_t\|_1 + \|\nabla u_t^\varepsilon - \nabla u_t\|_1, \\ &\leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} + \|u_t^\varepsilon - u_t\|_1 + \|\nabla u_t^\varepsilon - \nabla u_t\|_1, \end{aligned} \quad (5.34)$$

where we have used Proposition 5.2 for the second inequality above. Taking into account Theorem 4.5 above, it appears clearly that the convergence of $u^{\varepsilon, N}$ (resp. $\nabla u^{\varepsilon, N}$) to u (resp. ∇u) will hold as soon as $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \rightarrow 0$ when $\varepsilon \rightarrow 0, N \rightarrow +\infty$. This concludes the proof of (5.32).

The second inequality (5.33), concerning the specific case where Λ does not depend on the gradient ∇u , is proved similarly by gathering inequality (4.38) from Proposition 4.7 and inequality (5.10) from Proposition 5.2. □

Remark 5.4. 1. *In the first statement of Corollary 5.3 appears the condition $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \rightarrow 0$ when $\varepsilon \rightarrow 0, N \rightarrow +\infty$. This requires a "trade-off" between the speed of convergence of N and ε . Setting $\Phi(\varepsilon) := \varepsilon^{-(d+4)} e^{\frac{2C}{\varepsilon^{d+1}}}$, the trade-off condition can be formulated as*

$$\frac{\Phi(\varepsilon)}{N} \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0, N \rightarrow +\infty. \quad (5.35)$$

An example of such trade-off between N and ε can be given by the relation $\varepsilon(N) = \left(\frac{1}{\log(N)} \right)^{\frac{1}{d+4}}$.

2. *The estimate (5.33) recovers the same order of convergence as the one encountered in classical density estimates, see e.g. (22) in [16]. This happens in spite of the fact the weights V_t in (5.1) depend on the whole past of the whole particle system.*

6 Appendix

6.1 General inequalities

If f is a probability density on \mathbb{R}^d , $I(f)$ denotes the quantity $I(f) := \int_{\mathbb{R}^d} |x|^{d+1} f(x) dx$.

Lemma 6.1 (Multidimensional Carlson's inequality). *Let f be a probability density on \mathbb{R}^d such that $I(f) < \infty$, then*

$$\int_{\mathbb{R}^d} \sqrt{f(x)} dx \leq A_d I(f)^{\frac{d}{2(d+1)}} \quad \text{where} \quad A_d = \left(\frac{2\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d}{2}) d^{\frac{d}{d+1}} \sin\left(\frac{d\pi}{d+1}\right)} \right)^{1/2}. \quad (6.1)$$

Proof. We apply (16) in Lemma 7 (p.251) of [16] setting $g = \sqrt{f}$, $\varepsilon = 1$. \square

From Lemma 6.1, we deduce the following lemma.

Lemma 6.2. *Let G and f be two probability densities defined on \mathbb{R}^d such that*

$$I(G) < \infty, \quad \text{and} \quad I(f) < \infty. \quad (6.2)$$

Then for any strictly positive real $\varepsilon \leq (1/I(G))^{\frac{1}{d+1}}$,

$$\int_{\mathbb{R}^d} \sqrt{(G_\varepsilon * f)(x)} dx \leq 2^{\frac{d}{2}} A_d [1 + I(f)], \quad \text{where} \quad A_d = \left(\frac{2\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d}{2}) d^{\frac{d}{d+1}} \sin\left(\frac{d\pi}{d+1}\right)} \right)^{1/2}, \quad (6.3)$$

and $G_\varepsilon(\cdot) := \frac{1}{\varepsilon^d} G(\frac{\cdot}{\varepsilon})$.

Proof. By Carlson's inequality (6.1) we have

$$\int_{\mathbb{R}^d} \sqrt{(G_\varepsilon * f)(x)} dx \leq A_d [I(G_\varepsilon * f)]^{\frac{d}{2(d+1)}}. \quad (6.4)$$

Then

$$\begin{aligned} [I(G_\varepsilon * f)]^{\frac{1}{d+1}} &= \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{d+1} G_\varepsilon(x-y) f(y) dy dx \right]^{\frac{1}{d+1}} \\ &= \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |y + u\varepsilon|^{d+1} G(u) f(y) dy du \right]^{\frac{1}{d+1}} \\ &\leq \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |y|^{d+1} G(u) f(y) dy du \right]^{\frac{1}{d+1}} + \varepsilon \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |u|^{d+1} G(u) f(y) dy du \right]^{\frac{1}{d+1}} \\ &\leq I(f)^{\frac{1}{d+1}} + \varepsilon I(G)^{\frac{1}{d+1}}. \end{aligned}$$

Since $x \in \mathbb{R}^+ \mapsto x^d$ is convex, it follows

$$I(G_\varepsilon * f)^{\frac{d}{2(d+1)}} \leq 2^{\frac{d-1}{2}} \left[[I(f)]^{\frac{d}{d+1}} + \varepsilon^d [I(G)]^{\frac{d}{d+1}} \right]^{\frac{1}{2}}.$$

Hence, as soon as $\varepsilon \leq (1/I(G))^{\frac{1}{d+1}}$, we have

$$[I(G_\varepsilon * f)]^{\frac{d}{2(d+1)}} \leq 2^{\frac{d}{2}} [1 + I(f)], \quad (6.5)$$

which, owing to (6.4), concludes the proof. \square

Lemma 6.3. *Let H be a density kernel on \mathbb{R}^d satisfying*

$$H \geq 0, \quad \int_{\mathbb{R}^d} H(x) dx = 1. \quad (6.6)$$

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a real-valued function. For any $\varepsilon > 0$, we consider the function H_ε given by

$$H_\varepsilon(\cdot) := \frac{1}{\varepsilon^d} H\left(\frac{\cdot}{\varepsilon}\right). \quad (6.7)$$

If $a := \frac{1}{2} \int_{\mathbb{R}^d} |x| H(x) dx < \infty$ $f \in W^{1,p}$ for some integer $p \geq 1$, then for any $\varepsilon > 0$,

$$\|H_\varepsilon * f - f\|_p \leq \varepsilon a \sum_{i=1}^d \|\partial_i f\|_p. \quad (6.8)$$

Proof. The proof is modeled on [16]. For $\varepsilon > 0$ and any integer $1 \leq i \leq d$, let us introduce the real-valued function L_ε^i defined on \mathbb{R}^d with values in $\bar{\mathbb{R}}_+$, associated with H such that for almost all $x \in \mathbb{R}^d$,

$$L_\varepsilon^i(x) = \frac{x_i}{\varepsilon} \int_0^1 \frac{1-t}{t} H_{\varepsilon t}(x) dt, \quad (6.9)$$

where x_i is the i -th coordinate of x and H_t given by (6.7). Observe that, for any $\varepsilon > 0$, $1 \leq i \leq d$,

$$\sum_{i=1}^d \|L_\varepsilon^i\|_1 = \int_{\mathbb{R}^d} \sum_{i=1}^d |L_\varepsilon^i(x)| dx = a, \quad (6.10)$$

which implies that $L_\varepsilon^i < \infty$ a.e. Developing f according to the Lagrange expansion up to order one, yields, for almost all $(x, y) \in (\mathbb{R}^d)^2$,

$$f(x-y) = f(x) - \sum_{i=1}^d \int_0^1 (1-t) (\partial_i f)(x-ty) y_i dt.$$

Integrating this expression against H_ε w.r.t. y and using the symmetry of H , yields for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned} (H_\varepsilon * f)(x) - f(x) &= \int_{\mathbb{R}^d} [f(x-y) - f(x)] H_\varepsilon(y) dy \\ &= \sum_{i=1}^d \int_{\mathbb{R}^d} \int_0^1 (1-t) \partial_i f(x-ty) y_i dt H_\varepsilon(y) dy \\ &= \varepsilon \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_i f(x-u) \frac{u_i}{\varepsilon} \int_0^1 \frac{1-t}{t} H_{\varepsilon t}(u) dt du \\ &= \varepsilon \sum_{i=1}^d (L_\varepsilon^i * (\partial_i f))(x). \end{aligned} \quad (6.11)$$

Taking the L^p norm in equality (6.11), Young's inequality yields

$$\|H_\varepsilon * f - f\|_p \leq \varepsilon \sum_{i=1}^d \|\partial_i f\|_p \|L_\varepsilon^i\|_1,$$

which gives the result by recalling (6.10). □

6.2 About transition kernels

In the following lemma, we state well-known technical properties about the transition probability function of a diffusion process. All the statements below are established in [12].

Lemma 6.4. *We assume here the validity of items 1. to 3. of Assumption 2. Consider a stochastic process Y , solution of the SDE*

$$Y_t = Y_0 + \int_0^t \Phi(s, Y_s) dW_s + \int_0^t g(s, Y_s) ds. \quad (6.12)$$

$P(s, x_0, t, \Gamma)$ denotes its transition probability function, for all $(s, x_0, t, \Gamma) \in [0, T] \times \mathbb{R}^d \times [0, T] \times \mathcal{B}(\mathbb{R}^d)$. The following statements hold.

1. The transition probability function P admits a density, i.e. there exists a Borel function $p : (s, x_0, t, x) \mapsto p(s, x_0, t, x)$ such that for all $(s, x_0, t) \in [0, T] \times \mathbb{R}^d \times [0, T]$ with $s < t$,

$$P(s, x_0, t, \Gamma) = \int_{\Gamma} p(s, x_0, t, x) dx \quad , \quad \Gamma \in \mathcal{B}(\mathbb{R}^d) . \quad (6.13)$$

2. The partial derivatives of the map $x_0 \mapsto p(s, x_0, t, x)$ exist in the distributional sense. For almost all $0 \leq s < t \leq T$ and $x_0, x \in \mathbb{R}^d$ there are constants $C_u, c_u > 0$, only depending on Φ, g such that

$$p(s, x_0, t, x) \leq C_u q(s, x_0, t, x) \quad (6.14)$$

and

$$|\partial_{x_0} p(s, x_0, t, x)| \leq C_u \frac{1}{\sqrt{t-s}} q(s, x_0, t, x) , \quad (6.15)$$

where $q(s, x_0, t, x) := \left(\frac{c_u(t-s)}{\pi} \right)^{\frac{d}{2}} e^{-c_u \frac{|x-x_0|^2}{t-s}}$ is a Gaussian kernel.

In particular for all $t \in [0, T]$ for almost all r, x we have

$$\sup_x \int p(r, x_0, t, x) dx_0 \leq C_u . \quad (6.16)$$

Proof. See Theorem 5.4, Section 5 in Chapter 6 in [12], Section 4 of [12], the fact that classical solutions of (2.15) are also distributional solutions together with Theorem 15, Section 9, chap. 1 in [11] and inequalities (8.13) and (8.14) just before. \square

6.3 Proof of Proposition 2.2

For given u we set $\hat{\Lambda}(s, x) := \Lambda(s, u(s, x), \nabla u(s, x))u(s, x)$. We first suppose that u is a mild solution of (1.1). Taking into account that $P(s, x_0, t, \cdot)$ is a distributional solution of (2.18), we show below that u is indeed a weak solution of (1.1).

Indeed, for $0 \leq r < t \leq T$, (2.16), gives

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) P(r, x_0, t, dx) \mathbf{u}_0(dx_0) &= \int_r^t \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} L_s \varphi(x) P(r, x_0, s, dx) \mathbf{u}_0(dx_0) \right) ds \\ &+ \int_{\mathbb{R}^d} \varphi(x_0) \mathbf{u}_0(dx_0), \quad \forall \varphi \in C_0^\infty . \end{aligned} \quad (6.17)$$

For every $s \in [0, T]$ we define the measure

$$v(s, dx) = \int_{\mathbb{R}^d} P(0, x_0, s, dx) \mathbf{u}_0(dx_0) + \int_0^s \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) P(r, x_0, s, dx) dx_0 dr . \quad (6.18)$$

Since u is a mild solution of (1.1), see (2.20), we have

$$u(s, x) dx = v(s, dx) .$$

In particular $v(s, \cdot)$ admits $u(s, \cdot)$ as density. We need to show that for all $\varphi \in C_0^\infty$, $t \in [0, T]$ we have (2.19), i.e. for any test function $\varphi \in C_0^\infty(\mathbb{R}^d)$ and any $t \in [0, T]$

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} v(s, dx) L_s \varphi(x) dx ds &= \int_{\mathbb{R}^d} \varphi(x) v(t, dx) - \int_{\mathbb{R}^d} \varphi(x) \mathbf{u}_0(dx) \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \varphi(x) \hat{\Lambda}(s, x) dx ds. \end{aligned} \quad (6.19)$$

Starting with the left-hand side of (6.19), we start by plugging in the expression of v in (6.18) into the right-hand side of (6.19):

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} L_s \varphi(x) v(s, dx) ds &= \int_0^t \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) \int_{\mathbb{R}^d} L_s \varphi(x) P(0, x_0, s, dx) ds \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^d} \int_0^s \hat{\Lambda}(r, x_0) \int_{\mathbb{R}^d} L_s \varphi(x) P(r, x_0, s, dx) dr dx_0 \right) ds. \end{aligned} \quad (6.20)$$

For the first term on the right-hand side of (6.20), we can directly use identity (6.17) to infer

$$\begin{aligned} &\int_0^t \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} L_s \varphi(x) P(0, x_0, s, dx) \right) \mathbf{u}_0(dx_0) \right) ds \\ &= \int_{\mathbb{R}^d} \mathbf{u}_0(dx_0) \left(\int_{\mathbb{R}^d} \varphi(x) P(0, x_0, t, dx) \right) - \int_{\mathbb{R}^d} \varphi(x_0) \mathbf{u}_0(dx_0). \end{aligned} \quad (6.21)$$

For the second term on the right-hand side, a simple application of Fubini's Theorem for random kernels enables us to "pull out" the integral with respect to r and then apply (6.17):

$$\begin{aligned} &\int_0^t \left(\int_{\mathbb{R}^d} \int_0^s \hat{\Lambda}(r, x_0) \int_{\mathbb{R}^d} L_s \varphi(x) P(r, x_0, s, dx) dx_0 dr \right) ds \\ &= \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) \left(\int_r^t \int_{\mathbb{R}^d} P(r, x_0, s, dx) L_s \varphi(x) ds \right) dx_0 dr \\ &= \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) \left(\int_{\mathbb{R}^d} \varphi(x) P(r, x_0, t, dx) - \varphi(x_0) \right) dx_0 dr. \end{aligned} \quad (6.22)$$

Plugging the equalities (6.21), (6.22), into (6.20) leaves us with

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} v(r, dx) L_s \varphi(x) dr \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) P(0, x_0, t, dx) \mathbf{u}_0(dx_0) - \int_{\mathbb{R}^d} \varphi(x) \mathbf{u}_0(dx) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \hat{\Lambda}(r, x_0) \left(\int_{\mathbb{R}^d} \varphi(x) P(s, x_0, r, dx) \right) dx_0 dr - \int_{\mathbb{R}^d} \varphi(x) \hat{\Lambda}(r, x) dx dr \\
&= \int_{\mathbb{R}^d} \varphi(x) v(t, dx) - \int_{\mathbb{R}^d} \varphi(x) \mathbf{u}_0(dx) - \int_0^t \int_{\mathbb{R}^d} \varphi(x) \hat{\Lambda}(s, x) dx ds,
\end{aligned}$$

where for the latter equality we again used the definition of v in (6.18). This is exactly (6.19) and thus completes the first part of the proof.

Conversely, suppose that u is a weak solution of (1.1), in the sense of Definition 2.1. We also consider

$$\begin{aligned}
\bar{v}(t, dx) &:= \int_{\mathbb{R}} P(0, x_0, t, dx) \mathbf{u}_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) u(s, dx_0) \hat{\Lambda}(s, x_0) \\
&= \int_{\mathbb{R}} P(0, x_0, t, dx) \mathbf{u}_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \hat{\Lambda}(u)(s, x_0) dx_0.
\end{aligned} \tag{6.23}$$

We want to ensure that $u = \bar{v}$. On the one hand, by the first part of the proof applied to \bar{v} instead of v we can show that

$$\begin{cases} \partial_t \bar{v} = L_t^* \bar{v} + \hat{\Lambda} \\ \bar{v}(0, \cdot) = \mathbf{u}_0. \end{cases} \tag{6.24}$$

On the other hand, u being a weak solution of (1.1), it also a solution of (6.24) in the sense of distributions. We set $w := \bar{v} - u$. It follows that w and the zero measure function $\bar{w} \equiv 0$ $\hat{v} := 0$ both satisfy (2.15) in the sense of distributions, see (2.16). Uniqueness of the solution of (2.15) implies that $w = 0$, which concludes the proof.

6.4 Proof of technicalities of Section 3

We give in this section the proof of Lemma 3.4.

Proof of Lemma 3.4. We only prove the direct implication since the converse follows easier with similar arguments. The aim is to prove, for all $n \in \{1, \dots, N\}$,

$$(H_n) \begin{cases} \mu(t, dx) = \int_{\mathbb{R}^d} P(0, x_0, t, dx) \mathbf{u}_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0), \\ \text{for all } t \in [0, n\tau]. \end{cases} \tag{6.25}$$

We are going to proceed by induction on n . For $n = 1$, formula (6.25) follows from (3.13) by taking $k = 0$. We suppose now that (H_{n-1}) holds for some integer $n \geq 1$. Then, by taking $t = (n-1)\tau$ in the first line equation of (6.25), it follows immediately that

$$\mu((n-1)\tau, dx_0) = \int_{\mathbb{R}^d} P(0, \tilde{x}_0, (n-1)\tau, dx_0) \mathbf{u}_0(d\tilde{x}_0) + \int_0^{(n-1)\tau} ds \int_{\mathbb{R}^d} P(s, \tilde{x}_0, (n-1)\tau, dx_0) \tilde{\Lambda}(s, \tilde{x}_0) \mu(s, d\tilde{x}_0). \tag{6.26}$$

On the other hand, since (3.13) is valid for all $t \in [(n-1)\tau, n\tau]$ by plugging $k = n-1$, we obtain

$$\mu(t, dx) = \int_{\mathbb{R}^d} P((n-1)\tau, x_0, t, dx) \mu((n-1)\tau, dx_0) + \int_{(n-1)\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0), \quad (6.27)$$

for all $t \in [(n-1)\tau, n\tau]$. Inserting (6.26) in (6.27) yields

$$\begin{aligned} \mu(t, dx) &= \int_{\mathbb{R}^d} \mathbf{u}_0(d\tilde{x}_0) \int_{\mathbb{R}^d} P(0, \tilde{x}_0, (n-1)\tau, dx_0) P((n-1)\tau, x_0, t, dx) \\ &\quad + \int_0^{(n-1)\tau} ds \int_{\mathbb{R}^d} \mu(s, d\tilde{x}_0) \tilde{\Lambda}(s, \tilde{x}_0) \int_{\mathbb{R}^d} P(s, \tilde{x}_0, (n-1)\tau, dx_0) P((n-1)\tau, x_0, t, dx) \\ &\quad + \int_{(n-1)\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0), \quad t \in [(n-1)\tau, n\tau]. \end{aligned} \quad (6.28)$$

Invoking the Chapman-Kolmogorov equation satisfied by the transition probability function $P(s, x_0, t, dx)$ (see e.g. expression (2.1) in Section 2.2, Chapter 2 in [31]), we have

$$P(s, \tilde{x}_0, t, dx) = \int_{\mathbb{R}^d} P(s, \tilde{x}_0, \theta, dz) P(\theta, z, t, dx), \quad s < \theta < t, (\tilde{x}_0, z) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (6.29)$$

Applying (6.29) with $\theta = (n-1)\tau$, it follows that for all $t \in [0, n\tau]$,

$$\begin{aligned} \mu(t, dx) &= \int_{\mathbb{R}^d} \mathbf{u}_0(d\tilde{x}_0) P(0, \tilde{x}_0, t, dx) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} P(s, \tilde{x}_0, t, dx) \tilde{\Lambda}(s, \tilde{x}_0) \mu(s, d\tilde{x}_0). \end{aligned} \quad (6.30)$$

This shows that (H_n) holds. □

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