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QUANTIFICATION OF THE UNIQUE CONTINUATION PROPERTY FOR THE HEAT EQUATION

Laurent Bourgeois

Laboratoire POEMS, Ensta ParisTech
828 Boulevard des Maréchaux
91120 Palaiseau, France

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Abstract. In this paper we prove a logarithmic stability estimate in the whole domain for the solution to the heat equation with a source term and lateral Cauchy data. We also prove its optimality up to the exponent of the logarithm and show an application to the identification of the initial condition and to the convergence rate of the quasi-reversibility method.

1. **Introduction.** This paper deals with the ill-posed problem of finding the solution to the heat equation from a source term and lateral Cauchy data. To be precise, let us denote by Ω a smooth domain of $\mathbb{R}^d$, $d \geq 1$, Γ_0 an open subset of $\partial\Omega$ and n the outward unit normal to Ω. Given $T > 0$, some function f in $\Omega \times (0, T)$ and two functions (g_0, g_1) on $\Gamma_0 \times (0, T)$, our problem consists in finding u in $\Omega \times (0, T)$ such that

$$\left\{ \begin{array}{rll} \partial_t u - \Delta u = f & \text{in} & \Omega \times (0,T) \\ u = g_0 & \text{on} & \Gamma_0 \times (0,T) \\ \partial_n u = g_1 & \text{on} & \Gamma_0 \times (0,T). \end{array} \right. \tag{1}$$

Problems such as (1) arise in the framework of inverse problems related to the heat equation, where $f = 0$ and the data (g_0, g_1) correspond to measurements on the accessible part Γ_0 of the boundary of Ω (see for example [6, 2]). The uniqueness property for that problem, that is $(f, g_0, g_1) = 0 \Rightarrow u = 0$, is well-known and is a consequence of Holmgren's Theorem (see for example Theorem 5.3.3 in [11]). However it is also well-known that this problem is ill-posed in the sense of Hadamard, due to the fact that the existence property does not hold, even for smooth data. More difficult is the question of stability, that is the quantification of the uniqueness property. This question for our problem is the following: if f, g_0 and g_1 are small (instead of being zero), let say smaller than δ, how small is u with respect to δ ? This question has been studied for a long time and is part of the monographs or review papers [16, 12, 23, 24], which all refer to a long series of contributions. In those contributions the authors obtain stability estimates which may have different forms and depend on the regularity assumptions on the domain Ω and on the functions f, g_0 and g_1. In particular, it is well-established that the solution u has a Hölder dependence on the data (f, g_0, g_1) in any subdomain $\Omega_\varepsilon \times (\varepsilon, T - \varepsilon)$ of $\Omega \times (0, T)$, where Ω_ε is the subset of Ω excluding the points such that their distance to the

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complementary part of Γ_0 in $\partial\Omega$ are smaller than $\varepsilon > 0$. In this vein, we refer for example to [23] (see Theorem 3.5.1) or to [24] (see Theorem 5.1), where the stability estimates are obtained by using a Carleman estimate applied to solutions to the heat equation, even if these two papers propose different ways to exploit such Carleman estimate. It is worth noting that in the result of [23], the time interval is not truncated near the final time T. These Hölder-type stability estimates imply the uniqueness property but are stronger since they quantify the unique continuation. In [13], a logarithmic stability estimate is obtained in the whole domain $\Omega \times (0,T)$ but only in the case when the support of the lateral Cauchy data $\Gamma_0 \times (0,T)$ is the whole lateral boundary $\partial\Omega \times (0,T)$. In other words, the objective of [13] amounts to recover the initial condition, since all the spatial boundary data are known. In [3], a logarithmic estimate is obtained in a subdomain $\Omega \times (\varepsilon, T-\varepsilon)$ when $\Gamma_0 \subsetneq \partial\Omega$ for the non-stationary Stokes system instead of the heat equation, which is a similar but more complicated case. Thus, the same kind of estimate in $\Omega \times (\varepsilon, T-\varepsilon)$ could be derived in the case of the heat equation by using the arguments of [3]. However, to the author's knowledge, a stability estimate for the heat equation in the whole domain $\Omega \times (0,T)$ with lateral Cauchy data supported in $\Gamma_0 \times (0,T)$ with $\Gamma_0 \subsetneq \partial\Omega$ is unknown.

In this paper we obtain such a stability estimate in the whole domain $\Omega \times (0,T)$ in the particular case when Ω and Γ_0 satisfy the following assumption.

Assumption 1.1. The domain Ω satisfies $\Omega = D \setminus \overline{O}$, where D is an open, bounded and connected domain of class C^2 and $O \Subset D$ is an open domain of class C^2 (not necessarily connected), while Γ_0 is either the interior boundary ∂O or the exterior boundary ∂D of Ω.

Let us denote $Q = \Omega \times (0,T)$, $\Sigma_0 = \Gamma_0 \times (0,T)$ and $P = \partial_t - \Delta$. We obtain the following logarithmic stability estimate.

Main Theorem. *For all $s \in (0,1)$, there exists a constant $C > 0$ such that for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$,*

$$||u||_{L^2(0,T;H^1(\Omega))} \leq C \frac{||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}}{\log^s\left(2 + \frac{||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}}{||Pu||_{L^2(Q)} + ||u||_{H^1(\Sigma_0)} + ||\partial_n u||_{L^2(\Sigma_0)}}\right)}.$$

As exposed in Remark 5, such theorem may be considered as an extension of the logarithmic stability estimates obtained for the Laplace equation in [20] (for a domain of class C^∞) and [4] (for a domain of class $C^{1,1}$). Other logarithmic estimates for the conductivity equation or/and non-smooth domains may be found in [1, 21, 5]. The proof of the Main Theorem consists of two steps. The first one is a stability estimate in the subdomain $\Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$ obtained with the help of a Carleman estimate. The second step consists in completing the previous estimate in the whole domain $\Omega \times (0,T)$ with the help of a Hardy-type inequality. Such general scheme was first introduced in [20] for the Laplace equation. The main issue in step 1 consists in exhibiting in the estimate the explicit dependence of the constants with respect to ε. To this aim we need to know the explicit dependence of the constants in the Carleman estimate with respect to the length of the time interval, as it is done in [7, 17]. To obtain the Carleman estimate, we proceed similarly as in the pioneering contribution [8], that is by using some Carleman weight that vanishes when $t(T-t) \to 0$. Such kind of weight is also used in [7, 17] and differs from the one used in [16, 24]. As in [24, 3], we apply the Carleman estimate in many time intervals covering $(0,T)$ up to the small parameter ε and the length of which tends to 0 when $\varepsilon \to 0$, but the novelty here consists in deriving an estimate in each interval, the constants of which depend explicitly on ε. Another novelty of this paper consists in constructing a smooth Carleman weight which close to boundary ∂D or ∂O coincides with the distance function to such boundary. The combination of such idea and the geometric assumption 1.1 considerably simplifies the proof. The possibility of getting rid of Assumption 1.1 is discussed at the end of the paper.

Our article is organized as follows. The second section is devoted to the construction of a useful Carleman weight with the help of a slight modification of Theorem 9.4.3 in [22]. The third one is devoted to the derivation of the Carleman estimate while the Main Theorem is proved in section 4 and its optimality discussed in section 5. We then show that the Main Theorem enables us to estimate the initial condition (section 6) and to obtain a convergence rate for the method of quasi-reversibility (section 7). We complete the paper by some remarks about the geometric assumption 1.1.

2. **Construction of a Carleman weight.** In order to derive a useful Carleman estimate, we will need the following theorem.

Theorem 2.1. *Let $\mathcal{D} \subset \mathbb{R}^d$ be an open bounded and connected domain of class C^m, with $m \geq 2$. Let $\mathcal{O}$ be an open subset of $\mathcal{D}$. There exists a function $\eta \in C^m(\overline{\mathcal{D}})$ such that*

- *$\eta(x) > 0$ for all $x \in \mathcal{D}$,*
- *η coincides with the distance function $d(\cdot, \partial\mathcal{D})$ in $\mathcal{V} = \overline{\mathcal{N}} \cap \overline{\mathcal{D}}$, where $\mathcal{N}$ is an open neighborhood of $\partial\mathcal{D}$ in $\mathbb{R}^d$,*
- *$|\nabla\eta(x)| > 0$ for all $x \in \overline{\mathcal{D} \setminus \mathcal{O}}$.*

Theorem 2.1 is a slight refinement of Theorem 9.4.3 in [22]: the only difference is that the second property in Theorem 2.1 is replaced by the weaker property $\eta(x) = 0$ for all $x \in \partial\mathcal{D}$. The proof of Theorem 9.4.3 in [22] contains two steps. The first one is a construction of a smooth positive function v which vanishes on the boundary of $\mathcal{D}$ and with no critical points in a tubular vicinity of the boundary of $\mathcal{D}$. The second step consists in moving the critical points of v into the domain $\mathcal{O}$ with the help of a well chosen map which preserves the other properties of the function v. The proof of Theorem 2.1 follows the same lines. The only difference with the proof of Theorem 9.4.3 in [22] (see Chapter 14) is the replacement of the so-called Step 1 in the proof of the intermediate Theorem 14.2.3 in [22] by the following lemma.

Lemma 2.2. *Let $\mathcal{D} \subset \mathbb{R}^d$ be an open bounded and connected domain of class C^m, with $m \geq 2$. There exists a function $v \in C^m(\overline{\mathcal{D}})$ such that*

- *$v(x) > 0$ for all $x \in \mathcal{D}$,*
- *v coincides with the distance function $d(\cdot, \partial\mathcal{D})$ in $\mathcal{V} = \overline{\mathcal{N}} \cap \overline{\mathcal{D}}$, where $\mathcal{N}$ is an open neighborhood of $\partial\mathcal{D}$ in $\mathbb{R}^d$,*
- *$|\nabla v(x)| > 0$ for all $x \in \mathcal{V}$.*

Proof. Let us denote, for $x \in \overline{\mathcal{D}}$, $d(x) = d(x, \partial\mathcal{D})$ the distance function to the boundary $\partial\mathcal{D}$. Such function is not smooth in $\mathcal{D}$ but from [9] (see Lemma 14.16), there exists an open neighborhood $\mathcal{N}_0$ of $\partial\mathcal{D}$ in $\mathbb{R}^d$ such that $d \in C^m(\mathcal{V}_0)$ where $\mathcal{V}_0 = \overline{\mathcal{N}_0} \cap \overline{\mathcal{D}}$ and $\nabla d(x) = -n(x')$ for $x \in \mathcal{V}_0$, where x' is the unique point in $\partial\mathcal{D}$ such that $d(x) = |x - x'|$ and n is the outward unit normal vector. In particular

$|\nabla d(x)| > 0$ for $x \in \mathcal{V}_0$. Now let us consider two smaller open neighborhoods $\mathcal{N} \Subset \mathcal{N}_1 \Subset \mathcal{N}_0$ of $\partial\mathcal{D}$ in $\mathbb{R}^d$ and $\mathcal{V} = \overline{\mathcal{N}} \cap \overline{\mathcal{D}}$, $\mathcal{V}_1 = \overline{\mathcal{N}_1} \cap \overline{\mathcal{D}}$. Let us introduce the cutoff function $\phi \in C_0^\infty(\mathcal{D})$ with $0 \leq \phi \leq 1$, $\phi = 1$ on the closure of $\mathcal{D} \setminus \mathcal{V}_1$ and $\phi = 0$ on $\mathcal{V}$. The function v defined in $\overline{\mathcal{D}}$ by

$$v = \phi + (1-\phi)d$$

satisfies all the requirements of the lemma. □

Classically, the function η derived in Theorem 2.1 for $m = 2$ is used to build an adapted weight for a Carleman inequality that will be then used to obtain some controllability results (see for example [7]).

3. **A uniform Carleman estimate.** For an open connected domain $\Omega \subset \mathbb{R}^d$ of class C^2, let us consider a function $\eta \in C^2(\overline{\Omega})$ such that $\eta(x) \geq 0$ and $|\nabla\eta(x)| \neq 0$ for $x \in \overline{\Omega}$. Let us define

$$\phi(x,t) = \frac{e^{\lambda(2L+\eta(x))} - e^{4\lambda L}}{\theta(t)}, \quad \xi(x,t) = \frac{e^{\lambda(2L+\eta(x))}}{\theta(t)},$$

with $L = ||\eta||_\infty$ and $\theta(t) = t(T-t)$. We denote $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. The aim of this section is to prove the following Carleman estimate in Q.

Theorem 3.1. *There exist some constants $\lambda_0, \rho_0 > 0$ and $C > 0$ such that for all T, for all $\lambda > \lambda_0$, $\rho > \rho_0$ and all $u \in C^2(\overline{Q})$,*

$$s^3\lambda^4 \int_Q \xi^3 u^2\, e^{2s\phi}\, dxdt + s\lambda^2 \int_Q \xi|\nabla u|^2\, e^{2s\phi}\, dxdt \leq C \int_Q (Pu)^2\, e^{2s\phi}\, dxdt$$
$$+C\, s^3\lambda^3 \int_\Sigma \xi^3 u^2\, e^{2s\phi}\, dsdt + C\, s\lambda \int_\Sigma \xi|\nabla u|^2\, e^{2s\phi}\, dsdt + C\,\frac{1}{s\lambda}\int_\Sigma \frac{1}{\xi}(\partial_t u)^2\, e^{2s\phi}\, dsdt,$$

where $s = \rho(T+T^2)$.

Remark 1. It is important to note that in the Carleman estimate above the constants λ_0, ρ_0 and C are independent of T, this is why we call it a uniform Carleman estimate.

Proof of Theorem 3.1. We here reproduce the proof of Lemma 1.3 in [7] by using the same notations. However there are two differences: contrary to that lemma, on the one hand we do not assume that $u = 0$ on the boundary Σ and on the second hand we have $\nabla\eta \neq 0$ in the whole domain Ω instead of in a subdomain. The consequence is that the right-hand side of the Carleman estimate involves some boundary terms instead of a volume term in a subdomain. We will only insist on the boundary terms and refer to [7] for the remainder of the proof. Let us denote $w = e^{s\phi}u$, that is $u = e^{-s\phi}w$. We compute

$$\partial_t u = (\partial_t w - s\,\partial_t\phi\, w)e^{-s\phi},$$

while

$$\Delta u = (\Delta w + s^2|\nabla\phi|^2 w - 2s\,\nabla\phi\cdot\nabla w - s\,w\,\Delta\phi)e^{-s\phi}.$$

Given that

$$\nabla\phi = \lambda\xi\nabla\eta, \quad \Delta\phi = \lambda^2\xi^2|\nabla\eta|^2 + \lambda\xi\,\Delta\eta,$$

we have

$$e^{s\phi}Pu = \partial_t w - \Delta w + 2s\lambda\xi\,\nabla\eta\cdot\nabla w + \left(-s\partial_t\phi - s^2\lambda^2\xi^2|\nabla\eta|^2 + s\lambda^2\xi|\nabla\eta|^2 + s\lambda\xi\,\Delta\eta\right)w,$$

which we write

$$M_1 w + M_2 w = e^{s\phi}Pu + M_3 w,$$

with

$$M_1 w = 2s\lambda^2\xi|\nabla\eta|^2 w + 2s\lambda\xi\nabla\eta\cdot\nabla w + \partial_t w,$$

$$M_2 w = -s^2\lambda^2\xi^2|\nabla\eta|^2 w - \Delta w - s\partial_t\phi\, w$$

and

$$M_3 w = s\lambda^2\xi|\nabla\eta|^2 w - s\lambda\xi\,\Delta\eta\, w.$$

As a consequence, we have

$$(M_1 w, M_2 w)_{L^2(Q)} \leq ||e^{s\phi}Pu||^2_{L^2(Q)} + ||M_3 w||^2_{L^2(Q)}.$$

We observe that $M_1 w$ and $M_2 w$ are both sums of three terms, so we write with obvious notations

$$(M_1 w, M_2 w)_{L^2(Q)} = \sum_{i,j=1}^{3} I_{ij},$$

with

$$I_{11} = -2s^3\lambda^4\int_Q \xi^3|\nabla\eta|^4 w^2\,dxdt, \quad I_{21} = -2s^3\lambda^3\int_Q \xi^3|\nabla\eta|^2(\nabla\eta\cdot\nabla w)w\,dxdt,$$

$$I_{31} = -s^2\lambda^2\int_Q \xi^2|\nabla\eta|^2 w\,\partial_t w\,dxdt, \quad I_{12} = -2s\lambda^2\int_Q \xi|\nabla\eta|^2 w\,\Delta w\,dxdt,$$

$$I_{22} = -2s\lambda\int_Q \xi(\nabla\eta\cdot\nabla w)\Delta w\,dxdt, \quad I_{32} = -\int_Q \partial_t w\,\Delta w\,dxdt,$$

$$I_{13} = -2s^2\lambda^2\int_Q \xi|\nabla\eta|^2\partial_t\phi\, w^2\,dxdt, \quad I_{23} = -2s^2\lambda\int_Q \xi\partial_t\phi(\nabla\eta\cdot\nabla w)w\,dxdt,$$

$$I_{33} = -s\int_Q \partial_t\phi\, w\,\partial_t w\,dxdt.$$

After many applications of the Green Theorem and the fact that w and all its derivatives vanish at $t = 0$ and $t = T$ since $\phi \to -\infty$ for $x \in \Omega$ when $t \to 0^+$ or $t \to T^-$, we obtain

$$(M_1 w, M_2 w)_{L^2(Q)} = \int_Q V(w)\,dxdt - \int_\Sigma B(w)\,dsdt$$

by separating volume and boundary terms. While the terms I_{11}, I_{31}, I_{13}, I_{33} do not produce any boundary term (see equations (1.31), (1.34), (1.48) and (1.52) in [7]), the terms I_{21}, I_{12}, I_{22}, I_{32} and I_{23} produce the opposites of the following boundary terms (see equations (1.32), (1.37), (1.41), (1.45) and (1.49) in [7]):

$$B_{21} = s^3\lambda^3\int_\Sigma \xi^3|\nabla\eta|^2\partial_n\eta\, w^2\,dsdt, \quad B_{12} = 2s\lambda^2\int_\Sigma \xi|\nabla\eta|^2\partial_n w\, w\,dsdt,$$

$$B_{22} = 2s\lambda\int_\Sigma \xi(\nabla\eta\cdot\nabla w)\partial_n w\,dsdt, \quad B_{32} = \int_\Sigma \partial_t w\,\partial_n w\,dsdt,$$

$$B_{23} = s^2\lambda\int_\Sigma \xi\partial_t\phi\,\partial_n\eta\, w^2\,dsdt,$$

so that

$$\int_\Sigma B(w)\,dsdt = B_{21} + B_{12} + B_{22} + B_{32} + B_{23}.$$

Concerning the volume terms, it is proved in [7], more precisely in the Step 2 of the proof of Lemma 1.3, that there exist some constants $\lambda_0,\rho_0>0$ and $c>0$ such that for all T, for all $\lambda>\lambda_0$, $\rho>\rho_0$ and all $u\in C^2(\overline{Q})$,

$$\int_Q V(w)\,dxdt-||M_3w||^2_{L^2(Q)}\geq c\left(s^3\lambda^4\int_Q\xi^3w^2\,dxdt+s\lambda^2\int_Q\xi|\nabla w|^2\,dxdt\right).$$

In particular, this estimate uses the fact that $|\nabla\eta|\geq c$ in $\overline{\Omega}$ for some constant $c>0$. It remains to estimate the boundary terms.

$$|B_{21}|\leq C\,s^3\lambda^3\int_\Sigma\xi^3w^2\,dsdt,\quad |B_{22}|\leq C\,s\lambda\int_\Sigma\xi|\nabla w|^2\,dsdt.$$

By applying a Young's inequality to B_{12}, we obtain

$$|B_{12}|\leq s^2\lambda^3\int_\Sigma\xi^2w^2\,dsdt+\lambda\int_\Sigma|\nabla w|^2\,dsdt.$$

From the fact that $\xi\geq 4/T^2$ and $s\geq\rho T^2$, we obtain that $s\xi\geq 4\rho$, so that for large ρ,

$$|B_{12}|\leq C\,s^3\lambda^3\int_\Sigma\xi^3w^2\,dsdt+C\,s\lambda\int_\Sigma\xi|\nabla w|^2\,dsdt.$$

By applying now a Young's inequality to B_{32}, we obtain

$$|B_{32}|\leq s\lambda\int_\Sigma\xi|\nabla w|^2\,dsdt+\frac{1}{s\lambda}\int_\Sigma\frac{1}{\xi}\left|\partial_tw\right|^2\,dsdt.$$

Lastly, since

$$\partial_t\phi=(2t-T)\frac{e^{\lambda(2L+\eta)}-e^{4\lambda L}}{\theta^2(t)}$$

and

$$e^{4\lambda L}\leq e^{2\lambda(2L+\eta)},$$

we have

$$|\partial_t\phi|\leq T\xi^2. \tag{2}$$

As a consequence

$$|B_{23}|\leq s^2\lambda\int_\Sigma T\xi^3w^2\,dsdt$$

and since $s\geq\rho T$, we obtain for large ρ and large λ,

$$|B_{23}|\leq C\,s^3\lambda^3\int_\Sigma\xi^3w^2\,dsdt.$$

By gathering the above estimates we obtain that

$$\int_\Sigma B(w)\,dxdt\leq C\left(s^3\lambda^3\int_\Sigma\xi^3w^2\,dsdt+s\lambda\int_\Sigma\xi|\nabla w|^2\,dsdt+\frac{1}{s\lambda}\int_\Sigma\frac{1}{\xi}(\partial_tw)^2\,dsdt\right),$$

so that there exist some constants $\lambda_0,\rho_0>0$ and $C>0$ such that for all T, for all $\lambda>\lambda_0$, $\rho>\rho_0$ and all $u\in C^2(\overline{Q})$,

$$s^3\lambda^4\int_Q\xi^3w^2\,dxdt+s\lambda^2\int_Q\xi|\nabla w|^2\,dxdt\leq C\int_Q|Pu|^2e^{2s\phi}\,dxdt$$
$$+C\,s^3\lambda^3\int_\Sigma\xi^3w^2\,dsdt+C\,s\lambda\int_\Sigma\xi|\nabla w|^2\,dsdt+C\,\frac{1}{s\lambda}\int_\Sigma\frac{1}{\xi}(\partial_tw)^2\,dsdt.$$

To complete the proof we have to come back to the function u by using $w=ue^{s\phi}$, so that

$$\nabla w=(\nabla u+s\lambda\xi\nabla\eta\,u)e^{s\phi},\quad \partial_tw=(\partial_tu+s\partial_t\phi\,u)e^{s\phi}.$$

There exists some $c > 0$ such that

$$
\begin{aligned}
&s^3\lambda^4 \int_Q \xi^3 w^2 \, dxdt + s\lambda^2 \int_Q \xi |\nabla w|^2 \, dxdt \\
&\geq c\, s^3\lambda^4 \int_Q \xi^3 |\nabla \eta|^2 w^2 \, dxdt + c\, s\lambda^2 \int_Q \xi |\nabla w|^2 \, dxdt \\
&\geq c\, s^3\lambda^4 \int_Q \xi^3 |\nabla \eta|^2 u^2 e^{2s\phi} \, dxdt + c\, s\lambda^2 \int_Q \xi |\nabla u + s\lambda \xi \nabla \eta\, u|^2 e^{2s\phi} \, dxdt \\
&= 2c\, s^3\lambda^4 \int_Q \xi^3 |\nabla \eta|^2 u^2 e^{2s\phi} \, dxdt + c\, s\lambda^2 \int_Q \xi |\nabla u|^2 e^{2s\phi} \, dxdt \\
&+ 2c\, s^2\lambda^3 \int_Q \xi^2 (\nabla u \cdot \nabla \eta) u\, e^{2s\phi} \, dxdt.
\end{aligned}
$$

From the Young's inequality, for any $r \in (0,1)$,

$$
\begin{aligned}
&2\, s^2\lambda^3 \int_Q \xi^2 (\nabla u \cdot \nabla \eta) u e^{2s\phi} \, dxdt \\
&\leq \frac{1}{r} s^3\lambda^4 \int_Q \xi^3 |\nabla \eta|^2 u^2 e^{2s\phi} \, dxdt + r\, s\lambda^2 \int_Q \xi |\nabla u|^2 e^{2s\phi} \, dxdt,
\end{aligned}
$$

so that

$$
\begin{aligned}
&s^3\lambda^4 \int_Q \xi^3 w^2 \, dxdt + s\lambda^2 \int_Q \xi |\nabla w|^2 \, dxdt \\
&\geq c \left(2 - \frac{1}{r}\right) s^3\lambda^4 \int_Q \xi^3 |\nabla \eta|^2 u^2 e^{2s\phi} \, dxdt + c(1-r)\, s\lambda^2 \int_Q \xi |\nabla u|^2 e^{s\phi} \, dxdt.
\end{aligned}
$$

By choosing $r \in (1/2, 1)$ we obtain there exists $c > 0$ such that

$$
\begin{aligned}
&s^3\lambda^4 \int_Q \xi^3 w^2 \, dxdt + s\lambda^2 \int_Q \xi |\nabla w|^2 \, dxdt \\
&\geq c\, s^3\lambda^4 \int_Q \xi^3 u^2 e^{2s\phi} \, dxdt + c\, s\lambda^2 \int_Q \xi |\nabla u|^2 e^{2s\phi} \, dxdt.
\end{aligned}
$$

Besides, we have

$$
\begin{aligned}
&s^3\lambda^3 \int_\Sigma \xi^3 w^2 \, dsdt + s\lambda \int_\Sigma \xi |\nabla w|^2 \, dsdt \\
&= s^3\lambda^3 \int_\Sigma \xi^3 u^2 e^{2s\phi} \, dsdt + s\lambda \int_\Sigma \xi |\nabla u + s\lambda \xi \nabla \eta\, u|^2 e^{2s\phi} \, dsdt \\
&\leq C\, s^3\lambda^3 \int_\Sigma \xi^3 u^2 e^{2s\phi} \, dsdt + C\, s\lambda \int_\Sigma \xi |\nabla u|^2 e^{2s\phi} \, dsdt
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{s\lambda} \int_\Sigma \frac{1}{\xi} (\partial_t w)^2 \, dsdt = \frac{1}{s\lambda} \int_\Sigma \frac{1}{\xi} (\partial_t u + s\partial_t \phi\, u)^2 e^{2s\phi} \, dsdt \\
&\leq C \frac{1}{s\lambda} \int_\Sigma \frac{1}{\xi} (\partial_t u)^2 e^{2s\phi} \, dsdt + C \frac{s}{\lambda} \int_\Sigma \frac{1}{\xi} (\partial_t \phi)^2 u^2 e^{2s\phi} \, dsdt,
\end{aligned}
$$

which from (2) implies that

$$
\frac{1}{s\lambda} \int_\Sigma \frac{1}{\xi} (\partial_t w)^2 \, dsdt \leq C \frac{1}{s\lambda} \int_\Sigma \frac{1}{\xi} (\partial_t u)^2 e^{2s\phi} \, dsdt + C \frac{sT^2}{\lambda} \int_\Sigma \xi^3 u^2 e^{2s\phi} \, dsdt.
$$

From $s \geq \rho T$, we obtain for large ρ and large λ,

$$\frac{1}{s\lambda}\int_{\Sigma}\frac{1}{\xi}(\partial_t w)^2\,dsdt \leq C\,\frac{1}{s\lambda}\int_{\Sigma}\frac{1}{\xi}(\partial_t u)^2 e^{2s\phi}\,dsdt + C\,s^3\lambda^3\int_{\Sigma}\xi^3 u^2 e^{2s\phi}\,dsdt,$$

which completes the proof of the Theorem. □

4. **Proof of the Main Theorem.** Let us denote Γ_1 the complementary part of Γ_0 in $\partial\Omega$ and $\Sigma_1 = \Gamma_1 \times (0,T)$. As a consequence of the previous Carleman estimate, we will obtain an estimate in a subdomain of Q defined as the product of the space open domain Ω_ε defined by

$$\Omega_\varepsilon = \{x \in \Omega,\ d(x,\Gamma_1) > \varepsilon\}$$

by a time interval centered at an arbitrary point in $(0,T)$ and the length of which is of order $\varepsilon^{3/2}$.

Lemma 4.1. *There exist some positive constants $\nu, c, C_1, C_2, C, \rho_0, \varepsilon_0$ such that for all t_-, t_+ with $0 \leq t_- < t_+ \leq T$, for all $\rho > \rho_0$ and all $\varepsilon \in (0,\varepsilon_0)$, for all $u \in C^2(\overline{Q})$,*

$$||u||^2_{L^2(t^\varepsilon_-,t^\varepsilon_+;H^1(\Omega_\varepsilon))} \leq C\,e^{2C_1\rho(1+1/\tau)}\left(||Pu||^2_{L^2(q)} + ||u||^2_{H^1(\sigma_0)} + ||\partial_n u||^2_{L^2(\sigma_0)}\right) + C\,\varepsilon^{-2\nu}\,e^{-2C_2\rho\varepsilon(1+1/\tau)}\,||u||^2_{L^2(t_-,t_+;H^1(\Omega))},$$

with

$$q = \Omega \times (t_-,t_+), \quad \sigma_0 = \Gamma_0 \times (t_-,t_+),$$

$$t^\varepsilon_- = \frac{t_- + t_+}{2} - c\,\sqrt{\varepsilon}\,\frac{t_+ - t_-}{2}, \quad t^\varepsilon_+ = \frac{t_- + t_+}{2} + c\sqrt{\varepsilon}\,\frac{t_+ - t_-}{2},$$

and $\tau = t_+ - t_-$.

Proof. The proof is based on the Carleman estimate of Theorem 3.1 in the domain $q = \Omega \times (t_-,t_+)$. We also denote $\sigma = \partial\Omega \times (t_-,t_+)$. We recall that either $(\Gamma_0,\Gamma_1) = (\partial O, \partial D)$ or $(\Gamma_0,\Gamma_1) = (\partial D, \partial O)$. In order to specify the function η used in the Carleman estimate we have to consider these two cases separately in view of Assumption 1.1: we apply Theorem 2.1 for $m = 2$

- either with $\mathcal{D} = D$ and $\mathcal{O} = O$ if $(\Gamma_0,\Gamma_1) = (\partial O, \partial D)$,
- or with $\mathcal{D} = B \setminus \overline{O}$ and $\mathcal{O} = B \setminus \overline{D}$ if $(\Gamma_0,\Gamma_1) = (\partial D, \partial O)$, where B is an open ball of $\mathbb{R}^d$ such that $D \Subset B$.

In both cases we obtain a function $\eta \in C^2(\overline{\Omega})$ such that $\eta \geq 0$ in $\overline{\Omega}$, $\eta > 0$ in $\overline{\Omega}\setminus\Gamma_1$, $\nabla\eta \neq 0$ in $\overline{\Omega}$ and η coincides with the distance function to the boundary in a tubular vicinity of Γ_1. For any $u \in C^2(\overline{Q})$, let us define $v_\varepsilon = \chi_\varepsilon u$, where $\chi_\varepsilon \in C^2(\overline{\Omega})$ is a function of x with values in $[0,1]$ and such that, for sufficiently small ε

$$\begin{cases} \chi_\varepsilon(x) = 1 & \text{if} \quad d(x,\Gamma_1) \geq 2\varepsilon \\ \chi_\varepsilon(x) = 0 & \text{if} \quad d(x,\Gamma_1) \leq \varepsilon. \end{cases}$$

We can choose χ_ε such that

$$|\nabla\chi_\varepsilon(x)| \leq \frac{C}{\varepsilon}, \quad |\Delta\chi_\varepsilon(x)| \leq \frac{C}{\varepsilon^2}, \quad \forall x \in \overline{\Omega}.$$

To construct the function χ_ε it suffices to define a function $f_\varepsilon \in C^2(\mathbb{R})$ with values in $[0,1]$ such that

$$\begin{cases} f_\varepsilon(z) = 1 & \text{if} \quad z \geq 2\varepsilon \\ f_\varepsilon(z) = 0 & \text{if} \quad z \leq \varepsilon, \end{cases}$$

with $|f'_\varepsilon(z)| \leq C/\varepsilon$ and $|f''_\varepsilon(z)| \leq C/\varepsilon^2$, then to set $\chi_\varepsilon = f_\varepsilon \circ d(\cdot,\Gamma_1)$. Hence $v_\varepsilon \in C^2(\overline{Q})$ and vanishes in the vicinity of Σ_1.

We now apply Theorem 3.1 to the function v_ε in the domain $q = \Omega \times (t_-, t_+)$, with $\theta(t) = (t - t_-)(t_+ - t)$. For a fixed value of $\lambda > \lambda_0$, let us denote

$$g(z) = e^{4\lambda L} - e^{\lambda(2L+z)}, \quad z \in [0, L],$$

which is a positive and decreasing function.
Let us first consider the left-hand side of the Carleman estimate. For small ε and $x \in \overline{\Omega}_{3\varepsilon}$, we have $\eta(x) \geq 3\varepsilon$. Let us define $t_m = (t_- + t_+)/2$. For any $t_r \in (t_-, t_m)$, and $t \in (t_r, t_- + t_+ - t_r)$, $\theta(t) \geq \theta(t_r)$, that is

$$e^{s\phi(x,t)} = e^{-s\frac{g(\eta(x))}{\theta(t)}} \geq e^{-s\frac{g(3\varepsilon)}{\theta(t_r)}},$$

We have already seen that $s\xi \geq 1$ for large ρ, hence

$$\begin{aligned} &s^3\lambda^4 \int_{t_-}^{t_+} \int_\Omega \xi^3 v_\varepsilon^2 \, e^{2s\phi} \, dxdt + s\lambda^2 \int_{t_-}^{t_+} \int_\Omega \xi |\nabla v_\varepsilon|^2 \, e^{2s\phi} \, dxdt \\ &\geq e^{-2s\frac{g(3\varepsilon)}{\theta(t_r)}} \int_{t_r}^{t_+ + t_- - t_r} \int_{\Omega_{3\varepsilon}} (u^2 + |\nabla u|^2) \, dxdt. \end{aligned}$$

Let us now consider the right-hand side. We have

$$P(\chi_\varepsilon u) = \chi_\varepsilon \, Pu - 2\nabla\chi_\varepsilon \cdot \nabla u - \Delta\chi_\varepsilon \, u.$$

$$\begin{aligned} &\int_q (Pv_\varepsilon)^2 \, e^{2s\phi} \, dxdt \\ &\leq C \int_{t_-}^{t_+} \int_\Omega (Pu)^2 \, e^{2s\phi} \, dxdt + C\,\varepsilon^{-4} \int_{t_-}^{t_+} \int_{\Omega_\varepsilon \setminus \overline{\Omega}_{2\varepsilon}} (u^2 + |\nabla u|)^2 \, e^{2s\phi} \, dxdt. \end{aligned}$$

For $x \in \overline{\Omega}$, we have $0 \leq \eta(x) \leq L$. For $t \in (t_-, t_+)$, we have $\theta(t) \leq \theta(t_m)$, hence

$$e^{s\phi(x,t)} = e^{-s\frac{g(\eta(x))}{\theta(t)}} \leq e^{-s\frac{g(L)}{\theta(t_m)}}$$

and

$$\int_{t_-}^{t_+} \int_\Omega (Pu)^2 \, e^{2s\phi} \, dxdt \leq e^{-2s\frac{g(L)}{\theta(t_m)}} \int_{t_-}^{t_+} \int_\Omega (Pu)^2 \, dxdt.$$

For $x \in \overline{\Omega}_\varepsilon \setminus \Omega_{2\varepsilon}$, we have $0 \leq \eta(x) \leq 2\varepsilon$. For $t \in (t_-, t_+)$, we obtain

$$e^{s\phi(x,t)} = e^{-s\frac{g(\eta(x))}{\theta(t)}} \leq e^{-s\frac{g(2\varepsilon)}{\theta(t_m)}},$$

so that

$$\int_{t_-}^{t_+} \int_{\Omega_\varepsilon \setminus \overline{\Omega}_{2\varepsilon}} (u^2 + |\nabla u|^2) \, e^{2s\phi} \, dxdt \leq e^{-2s\frac{g(2\varepsilon)}{\theta(t_m)}} \int_{t_-}^{t_+} \int_\Omega (u^2 + |\nabla u|^2) \, dxdt.$$

We have, since $s\xi \geq 1$ for large ρ,

$$\begin{aligned} &s^3\lambda^3 \int_\sigma \xi^3 v_\varepsilon^2 \, e^{2s\phi} \, dsdt + s\lambda \int_\sigma \xi |\nabla v_\varepsilon|^2 \, e^{2s\phi} \, dsdt + \frac{1}{s\lambda} \int_\sigma \frac{1}{\xi} (\partial_t v_\varepsilon)^2 \, e^{2s\phi} \, dsdt \\ &\leq C \int_{\sigma_0} (s\xi)^3 (u^2 + |\nabla u|^2 + (\partial_t u)^2) e^{-2s\frac{g(L)}{\theta(t)}} \, dsdt \\ &\leq C \int_{\sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) f(s/\theta(t)) \, dsdt, \end{aligned}$$

with

$$f(z) = z^3 e^{-2g(L)z}.$$

Here we have used the fact that for fixed λ, $|\xi| \leq C/\theta$. It is not difficult to see that $f(z) \leq C\, e^{-g(L)z}$ for $z \geq 0$, so that

$$s^3\lambda^3 \int_\sigma \xi^3 v_\varepsilon^2\, e^{2s\phi}\, dsdt + s\lambda \int_\sigma \xi |\nabla v_\varepsilon|^2\, e^{2s\phi}\, dsdt + \frac{1}{s\lambda}\int_\sigma \frac{1}{\xi}(\partial_t v_\varepsilon)^2\, e^{2s\phi}\, dsdt$$
$$\leq C\, e^{-s\frac{g(L)}{\theta(t_m)}} \int_{\sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2)\, dsdt.$$

Gathering the previous estimates, we obtain that for large ρ,

$$e^{-2s\frac{g(3\varepsilon)}{\theta(t_r)}} \int_{t_r}^{t_+ + t_- - t_r} \int_{\Omega_{3\varepsilon}} (u^2 + |\nabla u|^2)\, dxdt$$
$$\leq C\, e^{-2s\frac{g(L)}{\theta(t_m)}} \int_{t_-}^{t_+}\int_\Omega (Pu)^2\, dxdt + C\,\varepsilon^{-4}\, e^{-2s\frac{g(2\varepsilon)}{\theta(t_m)}} \int_{t_-}^{t_+}\int_\Omega (u^2 + |\nabla u|^2)\, dxdt$$
$$+ C\, e^{-s\frac{g(L)}{\theta(t_m)}} \int_{\sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2)\, dsdt.$$

We finally obtain that for large ρ,

$$e^{-2s\frac{g(3\varepsilon)}{\theta(t_r)}} ||u||^2_{L^2(t_r, t_+ + t_- - t_r; H^1(\Omega_{3\varepsilon}))}$$
$$\leq C\, e^{-s\frac{g(L)}{\theta(t_m)}} \left(||Pu||^2_{L^2(q)} + ||u||^2_{H^1(\sigma_0)} + ||\partial_n u||^2_{L^2(\sigma_0)} \right)$$
$$+ C\,\varepsilon^{-4}\, e^{-2s\frac{g(2\varepsilon)}{\theta(t_m)}} ||u||^2_{L^2(t_-, t_+; H^1(\Omega))},$$

that is

$$||u||^2_{L^2(t_r, t_+ + t_- - t_r; H^1(\Omega_{3\varepsilon}))} \leq C\, e^{2s\,p_1} \left(||Pu||^2_{L^2(q)} + ||u||^2_{H^1(\sigma_0)} + ||\partial_n u||^2_{L^2(\sigma_0)} \right)$$
$$+ C\,\varepsilon^{-4}\, e^{-2s\,p_2}\, ||u||^2_{L^2(t_-, t_+; H^1(\Omega))},$$

with

$$p_1 = \frac{g(3\varepsilon)}{\theta(t_r)} - \frac{g(L)}{2\theta(t_m)}, \quad p_2 = \frac{g(2\varepsilon)}{\theta(t_m)} - \frac{g(3\varepsilon)}{\theta(t_r)}.$$

For all $\gamma \in (0,1)$ there is a unique $t_r \in (t_-, t_m)$ such that $\theta(t_r) = \gamma\,\theta(t_m) = \gamma\,\tau^2/4$. We have

$$p_1 \leq \frac{g(0)}{\theta(t_r)}$$

and

$$p_2 = \frac{g(2\varepsilon)}{\theta(t_r)} \left(\gamma - \frac{g(3\varepsilon)}{g(2\varepsilon)} \right).$$

The function $g(3\varepsilon)/g(2\varepsilon)$ is equal to $1 - c_0\varepsilon$ near $\varepsilon = 0$ at order 1. By choosing $\gamma = 1 - c_1\,\varepsilon$, with $0 < c_1 < c_0$ then for small ε there exist some constants $C_1, C_2 > 0$ such that

$$p_1 \leq \frac{C_1}{\tau^2}, \quad p_2 \geq \frac{C_2\,\varepsilon}{\tau^2},$$

for some constants $C_1, C_2 > 0$. The time t_r is obtained by solving the equation

$$(t_+ - t_r)(t_r - t_-) = \gamma \frac{(t_+ - t_-)^2}{4},$$

that is

$$t_r = \frac{t_- + t_+}{2} - \beta\,\frac{t_+ - t_-}{2}, \quad \beta = \sqrt{1-\gamma} = \sqrt{c_1}\sqrt{\varepsilon},$$

which is the result by setting $c = \sqrt{c_1/3}$ and setting $\varepsilon' = 3\varepsilon$. □

Then we gather the estimates obtained in Lemma 4.1 for a sequence of time intervals in order to obtain an estimate in the subdomain $\Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$ of Q.

Proposition 1. *There exist some positive constants $\nu, C_1, C_2, C, \rho_0, \varepsilon_0$ such that for all $\rho > \rho_0$ and all $\varepsilon \in (0, \varepsilon_0)$, for all $u \in C^2(\overline{Q})$,*

$$||u||_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_\varepsilon))} \leq C\, e^{C_1\rho/\varepsilon} \left(||Pu||_{L^2(Q)} + ||u||_{H^1(\Sigma_0)} + ||\partial_n u||_{L^2(\Sigma_0)}\right) + C\,\varepsilon^{-\nu}\, e^{-C_2\rho}\, ||u||_{L^2(0,T;H^1(\Omega))}.$$

Proof. Let us take $\gamma = 1 - c^2\varepsilon$, where c is as in the statement of Lemma 4.1, $\beta = \sqrt{1-\gamma} = c\sqrt{\varepsilon}$ and some $\tau > 0$. Let us define, for $n \in \mathbb{N}$ such that $\tau(1+n\beta) \leq T$, the sequences

$$\begin{cases} t_-^n = n\beta\tau \\ t_+^n = \tau + n\beta\tau \end{cases}$$

and

$$\begin{cases} \tilde{t}_-^n = \dfrac{\tau}{2} + (n - \dfrac{1}{2})\beta\tau \\ \tilde{t}_+^n = \dfrac{\tau}{2} + (n + \dfrac{1}{2})\beta\tau. \end{cases}$$

By using the notations of Lemma 4.1, for $t_- = t_-^n$ and $t_+ = t_+^n$, the corresponding values of t_-^ε and t_+^ε are

$$t_-^\varepsilon = \frac{t_-^n + t_+^n}{2} - c\sqrt{\varepsilon}\,\frac{t_+^n - t_-^n}{2}, \quad t_-^\varepsilon = \frac{\tau}{2} + n\beta\tau - \beta\frac{\tau}{2} = \tilde{t}_-^n$$

and

$$t_+^\varepsilon = \frac{t_-^n + t_+^n}{2} + c\sqrt{\varepsilon}\,\frac{t_+^n - t_-^n}{2}, \quad t_+^\varepsilon = \frac{\tau}{2} + n\beta\tau + \beta\frac{\tau}{2} = \tilde{t}_+^n.$$

Applying Lemma 4.1 we obtain that for large ρ and small ε, for all $u \in C^2(\overline{Q})$,

$$\begin{aligned} &||u||^2_{L^2(\tilde{t}_-^n,\tilde{t}_+^n;H^1(\Omega_\varepsilon))} \\ &\leq C\, e^{2C_1\rho(1+1/\tau)} \left(||Pu||^2_{L^2(t_-^n,t_+^n;L^2(\Omega))} + ||u||^2_{H^1(\sigma_0^n)} + ||\partial_n u||^2_{L^2(\sigma_0^n)}\right) \\ &+ C\,\varepsilon^{-2\nu}\, e^{-2C_2\rho\varepsilon(1+1/\tau)}\, ||u||^2_{L^2(t_-^n,t_+^n;H^1(\Omega))}, \end{aligned}$$

where $\sigma_0^n = \Gamma_0 \times (t_-^n, t_+^n)$. Let us consider N such that $\tau(1+N\beta) \leq T$. We have

$$\begin{aligned} &\sum_{n=0}^{N} ||u||^2_{L^2(\tilde{t}_-^n,\tilde{t}_+^n;H^1(\Omega_\varepsilon))} \\ &\leq C(N+1)\, e^{2C_1\rho(1+1/\tau)} \left(||Pu||^2_{L^2(Q)} + ||u||^2_{H^1(\Sigma_0)} + ||\partial_n u||^2_{L^2(\Sigma_0)}\right) \\ &+ C(N+1)\,\varepsilon^{-2\nu}\, e^{-2C_2\rho\varepsilon(1+1/\tau)}\, ||u||^2_{L^2(0,T;H^1(\Omega))}. \end{aligned}$$

It is readily seen that $\tilde{t}_+^n = \tilde{t}_-^{n+1}$, we hence end up with

$$\begin{aligned} &||u||^2_{L^2(\tilde{t}_-^0,\tilde{t}_+^N;H^1(\Omega_\varepsilon))} \\ &\leq C(N+1)\, e^{2C_1\rho(1+1/\tau)} \left(||Pu||^2_{L^2(Q)} + ||u||^2_{H^1(\Sigma_0)} + ||\partial_n u||^2_{L^2(\Sigma_0)}\right) \\ &+ C(N+1)\,\varepsilon^{-2\nu}\, e^{-2C_2\rho\varepsilon(1+1/\tau)}\, ||u||^2_{L^2(0,T;H^1(\Omega))}. \end{aligned}$$

Since

$$\tilde{t}_-^0 = \frac{\tau}{2}(1-\beta), \quad \tilde{t}_+^N = \frac{\tau}{2} + (N + \frac{1}{2})\beta\tau,$$

by choosing τ such that $\tau(1-\beta)/2=\varepsilon$, that is

$$\tau=\frac{2\varepsilon}{1-c\sqrt{\varepsilon}}$$

and then N such that

$$\tau(1+N\beta)\leq T,\quad \frac{\tau}{2}+(N+\frac{1}{2})\beta\tau\geq\frac{T}{2},$$

$$\begin{aligned}&||u||^2_{L^2(\varepsilon,T/2;H^1(\Omega_\varepsilon))}\\ &\leq C(N+1)\,e^{2C_1\rho(1+1/\tau)}\left(||Pu||^2_{L^2(0,T;L^2(\Omega))}+||u||^2_{H^1(\Sigma_0)}+||\partial_n u||^2_{L^2(\Sigma_0)}\right)\\ &+C(N+1)\,\varepsilon^{-2\nu}\,e^{-2C_2\rho\varepsilon(1+1/\tau)}\,||u||^2_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

We can make a symmetric construction with respect to $t=T/2$. In addition, we have $\tau\beta N\leq T$, hence there exists a constant C such that

$$N\leq\frac{C}{\varepsilon^{3/2}}.$$

We conclude that there exist some positive constants ν,C_1,C_2,C such that for large ρ and small ε, for all $u\in C^2(\overline{Q})$,

$$\begin{aligned}||u||^2_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_\varepsilon))}\leq C\,e^{2C_1\rho/\varepsilon}\left(||Pu||^2_{L^2(Q)}+||u||^2_{H^1(\Sigma_0)}+||\partial_n u||^2_{L^2(\Sigma_0)}\right)\\ +C\,\varepsilon^{-2\nu}\,e^{-2C_2\rho}\,||u||^2_{L^2(0,T;H^1(\Omega))},\end{aligned}$$

which completes the proof. □

Remark 2. If for a fixed arbitrary ε we optimize ρ in the estimate of Proposition 1, we obtain a standard Hölder stability estimate in $\Omega_\varepsilon\times(\varepsilon,T-\varepsilon)$ as in [24].

In what follows we will need the following Hardy-type inequality, that is

Lemma 4.2. *If $Q\subset\mathbb{R}^d$ is a bounded, connected and Lipschitz domain, and if $d_{\partial Q}(x)$ denotes the distance of x to ∂Q, then $\forall r\in(0,1/2)$, $\forall u\in H^r(Q)$,*

$$\left\|\frac{u}{d^r_{\partial Q}}\right\|_{L^2(Q)}\leq C\,||u||_{H^r(Q)},$$

with $C>0$ depending only on r and on Q.

Lemma 4.2 is for example proved in [10]. Next we obtain the following result, which combines Proposition 1 and Lemma 4.2. More precisely, the Hardy-type inequality of Lemma 4.2 enables us to extend the estimate of Proposition 1 in the subdomain $\Omega_\varepsilon\times(\varepsilon,T-\varepsilon)$ to the whole domain $Q=\Omega\times(0,T)$.

Proposition 2. *For all $s\in(0,1)$, there exist some constants $c,\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, for all $u\in H^1(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$,*

$$\begin{aligned}||u||_{L^2(0,T;H^1(\Omega))}\leq e^{c/\varepsilon}\left(||Pu||_{L^2(Q)}+||u||_{H^1(\Sigma_0)}+||\partial_n u||_{L^2(\Sigma_0)}\right)\\ +\varepsilon^s(||u||_{H^1(0,T;H^1(\Omega))}+||u||_{L^2(0,T;H^2(\Omega))}).\end{aligned}$$

Proof. We first specify ρ as a function of ε. For some $r\in(0,1)$, we set

$$\varepsilon^{-\nu}\,e^{-C_2\rho}=\varepsilon^r,$$

which implies that

$$e^{C_1\,\rho/\varepsilon} = \left(\frac{1}{\varepsilon^{(r+\nu)/C_2}}\right)^{C_1/\varepsilon} \leq e^{(c/\varepsilon)\log(1/\varepsilon)},$$

for some $c > 0$. By the Proposition 1 we conclude that for small ε, for all $u \in C^2(\overline{Q})$,

$$\begin{aligned} ||u||_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_\varepsilon))} &\leq C\,e^{(c/\varepsilon)\log(1/\varepsilon)}\big(||Pu||_{L^2(Q)} + ||u||_{H^1(\Sigma_0)} \\ &+ ||\partial_n u||_{L^2(\Sigma_0)}\big) + C\,\varepsilon^r\,||u||_{L^2(0,T;H^1(\Omega))}. \end{aligned} \tag{3}$$

Now let us use Lemma 4.2 in the (d+1)-dimensional domain $Q = \Omega \times (0,T)$. We have for all $r \in (0,1)$,

$$\left\|u/d_{\partial Q}^{r/2}\right\|_{L^2(Q)} \leq C\,||u||_{H^{r/2}(Q)} \leq C\,||u||_{H^{1/2}(Q)}.$$

In particular, by denoting $Q_\varepsilon = \Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$ and $R_\varepsilon = Q \setminus \overline{Q}_\varepsilon$, since $d_{\partial Q}(x,t) \leq \varepsilon$ for all $(x,t) \in R_\varepsilon$, we obtain

$$||u||_{L^2(R_\varepsilon)} \leq C\,\varepsilon^{r/2}||u||_{H^{1/2}(Q)} \leq C\,\varepsilon^{r/2}||u||_{L^2(Q)}^{1/2}||u||_{H^1(Q)}^{1/2},$$

which implies that for any $\eta > 0$,

$$||u||_{L^2(R_\varepsilon)} \leq C\,\eta\,||u||_{L^2(Q)} + \frac{\varepsilon^r}{\eta}||u||_{H^1(Q)}.$$

We notice that the domain R_ε is the union of three cylindrical domains, namely

$$R_\varepsilon = \Omega_\varepsilon \times (0,\varepsilon) \cup \Omega_\varepsilon \times (T-\varepsilon,T) \cup \omega_\varepsilon \times (0,T),$$

where $\omega_\varepsilon = \Omega \setminus \overline{\Omega}_\varepsilon$, so that by using the previous inequality for functions u and its first derivatives with respect to x_i, $i = 1, \cdots, d$, we end up with

$$\begin{aligned} &||u||_{L^2(0,\varepsilon;H^1(\Omega_\varepsilon))} + ||u||_{L^2(T-\varepsilon,T;H^1(\Omega_\varepsilon))} + ||u||_{L^2(0,T;H^1(\omega_\varepsilon))} \\ &\leq C\,\eta\,||u||_{L^2(0,T;H^1(\Omega))} + \frac{\varepsilon^r}{\eta}(||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}). \end{aligned} \tag{4}$$

Using the fact that

$$\begin{aligned} &||u||_{L^2(0,T;H^1(\Omega))} \\ &\leq ||u||_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_\varepsilon))} + ||u||_{L^2(0,\varepsilon;H^1(\Omega_\varepsilon))} \\ &+ ||u||_{L^2(T-\varepsilon,T;H^1(\Omega_\varepsilon))} + ||u||_{L^2(0,T;H^1(\omega_\varepsilon))}, \end{aligned}$$

by gathering the estimates (3) and (4) and by choosing η such that $C\,\eta = 1/2$, we end up with

$$\begin{aligned} ||u||_{L^2(0,T;H^1(\Omega))} \leq C\,e^{(c/\varepsilon)\log(1/\varepsilon)}&(||Pu||_{L^2(Q)} + ||u||_{H^1(\Sigma_0)} + ||\partial_n u||_{L^2(\Sigma_0)}) \\ &+ C\,\varepsilon^r(||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}). \end{aligned}$$

Now for any $a > 1$ we can choose c' such that for small ε,

$$e^{(c/\varepsilon)\log(1/\varepsilon)} \leq e^{c'/\varepsilon^a}.$$

Defining $\varepsilon' = \varepsilon^a$, we have that $\varepsilon^r = \varepsilon'^s$ with $s = r/a$. Since for any $s \in (0,1)$ we can choose some $r \in (0,1)$ and $a > 1$ such that $s = r/a$, this implies the result for any function in $C^2(\overline{Q})$. We conclude by density of $C^\infty(\overline{Q})$ in $H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$. □

We can now give the proof of the Main Theorem, which essentially consists of a classical optimization process in the estimate of Proposition 2 with respect to the small parameter ε.

Proof of the Main Theorem. Let us pick $s \in (0,1)$. By denoting $\delta = ||Pu||_{L^2(0,T;L^2(\Omega))} + ||u||_{H^1(\Sigma_0)} + ||\partial_n u||_{L^2(\Sigma_0)}$, $m = ||u||_{L^2(0,T;H^1(\Omega))}$ and $M = ||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}$, we have that for $\varepsilon \in (0, \varepsilon_0)$,

$$m \leq e^{c/\varepsilon}\delta + \varepsilon^s M. \tag{5}$$

Here c and ε_0 only depend on s. Denoting $f(\varepsilon) = e^{c/\varepsilon}\delta + \varepsilon^s M$ for $\varepsilon > 0$, the minimizer ε_m of f solves

$$g(\varepsilon_m) = \frac{M}{\delta}, \quad g(\varepsilon) := \frac{c}{s}\frac{1}{\varepsilon^{s+1}}e^{c/\varepsilon}.$$

The function g is decreasing with $g(0^+) = +\infty$ and $g(+\infty) = 0$, so that the above equation has a unique solution ε_m for each $M, \delta > 0$.
We treat two cases separately.

- If $\varepsilon_0 > \varepsilon_m$, then by choosing $\varepsilon = \varepsilon_m$ in (5) we obtain that

  $$m \leq (\frac{s}{c}\varepsilon_0 + 1)M\varepsilon_m^s = CM\varepsilon_m^s. \tag{6}$$

  For sufficiently large M/δ, ε_m is sufficiently small to have for some $c' > c$,

  $$\frac{M}{\delta} = g(\varepsilon_m) \leq e^{c'/\varepsilon_m}.$$

  It follows that $\varepsilon_m \leq c'/\log(M/\delta)$, and by plugging this estimate in (6) we obtain that for sufficiently large M/δ

  $$m \leq C\,\frac{M}{\log^s(M/\delta)}. \tag{7}$$

- If $\varepsilon_0 \leq \varepsilon_m$, we obtain $g(\varepsilon_0) \geq M/\delta$, and thus

  $$m \leq M \leq g(\varepsilon_0)\delta = C\frac{M}{M/\delta}.$$

  For large M/δ we have $M/\delta \geq \log^s(M/\delta)$, which implies again (7).

We conclude that the estimate (7) is true for all u such that M/δ is sufficiently large. This implies, up to a change of constant C, that

$$m \leq C\,\frac{M}{\log^s(2 + M/\delta)}$$

as soon as M/δ is sufficiently large. But since $m \leq M$, such an estimate is true for any M/δ, up to a change of constant C. □

Remark 3. Another presentation of the Main Theorem is the following: for all $s \in (0,1)$, there exists a constant $C > 0$ such that for all $M, \delta > 0$, for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ such that

$$||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))} \leq M,\ ||Pu||_{L^2(Q)} + ||u||_{H^1(\Sigma_0)} + ||\partial_n u||_{L^2(\Sigma_0)} \leq \delta,$$

then

$$||u||_{L^2(0,T;H^1(\Omega))} \leq C\frac{M}{\log^s(2 + M/\delta)}.$$

Remark 4. The Main Theorem can be extended with no additional difficulty to the case when the operator $P = \partial_t - \Delta$ is replaced by $P = \partial_t - \mathrm{div}(\sigma\nabla) + \boldsymbol{B}\cdot\nabla + a$ with $\sigma \in W^{1,\infty}(\Omega)$, $\boldsymbol{B} \in (L^\infty(\Omega))^d$, $a \in L^\infty(\Omega)$ and $\sigma(x) \geq c > 0$ for all $x \in \overline{\Omega}$.

Remark 5. It is readily seen that for a stationary solution u, that is when u is independent of time t, the Main Theorem reduces to: for all $s \in (0,1)$, there exists a constant $C > 0$ such that for all $u \in H^2(\Omega)$,

$$||u||_{H^1(\Omega)} \leq C \frac{||u||_{H^2(\Omega)}}{\log^s\left(2 + \frac{||u||_{H^2(\Omega)}}{||\Delta u||_{L^2(\Omega)} + ||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)}}\right)},$$

which is the result already obtained in [20, 4] for the Laplace operator. Hence our Main Theorem can be seen as a generalization of the result in the stationary case. In addition, under the restriction of Assumption 1.1, such result for the Laplacian could be obtained more easily than in [20, 4] by using a Carleman weight independent of time and based on the same function $\eta(x)$ as in the proof of the non stationary case.

5. **Optimality of our stability estimate.** In this section we discuss the optimality of the estimate given by the Main Theorem. For sake of simplicity we consider the 1D case and a particular 2D case. They correspond to the following geometries.

- In the 1D case, $Q = (0,1) \times (0,T)$, $\Sigma_0 = \{0\} \times (0,T)$, $\Sigma_1 = \{1\} \times (0,T)$.
- In the 2D case, $Q = (B(O,2) \setminus \overline{B(O,1)}) \times (0,T)$, $\Sigma_0 = C(O,1) \times (0,T)$, $\Sigma_1 = C(O,2) \times (0,T)$, where $B(O,r)$ (resp. $C(O,r)$) is the open ball (resp. the circle) of center O and radius r, with $r = 1,2$.

We have the following result, which proves that our logarithmic stability estimate is optimal, up to the exponent of the log. More precisely, we have proved in the Main Theorem that our estimate holds with exponent $s = 1 - \varepsilon$, while we establish now that this exponent cannot be larger than 2 in 1D and cannot be larger than 1 in 2D.

Theorem 5.1. *Let us consider one of the 1D or 2D geometries above. Assume there exists a non decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$,*

$$||u||_{L^2(0,T;H^1(\Omega))} \leq \frac{||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}}{f\left(\frac{||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}}{||Pu||_{L^2(Q)} + ||u||_{H^1(\Sigma_0)} + ||\partial_n u||_{L^2(\Sigma_0)}}\right)}.$$

Then there exists some constant C such that for large A,

$$f(A) \leq C \log^s(A),$$

where $s = 2$ in 1D and $s = 1$ in 2D.

Proof. We first focus on the 1D case. Let us consider for all $a > 0$ the smooth functions defined on $[0,1] \times [0,T]$ by

$$u_a(x,t) = \chi(x) v_a(x,t), \quad v_a(x,t) = e^{ax + a^2 t},$$

where χ is a smooth function on the segment $[0,1]$ such that $\chi(0) = \chi'(0) = 0$, $\chi(x) \in [0,1]$ for all $x \in [0,1]$ and $\chi(x) = 1$ for all $x \in [X,1]$ for some $X \in (0,1)$. Let us now denote

$$m_a = ||u_a||_{L^2(0,T;H^1(0,1))}, \quad M_a = ||u_a||_{H^1(0,T;H^1(0,1))} + ||u_a||_{L^2(0,T;H^2(0,1))}$$

and

$$\delta_a = ||Pu_a||_{L^2(Q)} + ||u_a||_{H^1(\Sigma_0)} + ||\partial_x u_a||_{L^2(\Sigma_0)}.$$

We hence have, for all $a > 0$,

$$m_a \leq \frac{M_a}{f(M_a/\delta_a)}. \tag{8}$$

It is important to note that the functions v_a satisfy the heat equation $Pv_a = 0$ in $(0,1) \times (0,T)$ and that, due to the boundary conditions satisfied by χ at $x = 0$, we have $u_a = 0$ and $\partial_x u_a = 0$ on Σ_0. We hence have

$$\delta_a = ||Pu_a||_{L^2(Q)} \quad \text{with} \quad Pu_a = -2\chi' \partial_x v_a - \chi'' v_a.$$

The idea is now to minorate the left-hand side of (8) and to majorate the right-hand side. As concerns the left-hand side we have

$$\begin{aligned} m_a^2 = \int_0^T \int_0^1 (u_a^2 + (\partial_x u_a)^2)\, dxdt &\geq \int_0^T \int_X^1 (v_a^2 + (\partial_x v_a)^2)\, dxdt \\ &= \int_0^T \int_X^1 (1+a^2) e^{2ax+2a^2t}\, dxdt \\ &= (1+a^2) \frac{e^{2a^2T} - 1}{2a^2} \frac{e^{2a} - e^{2aX}}{2a}. \end{aligned}$$

Hence there exists some constant $c > 0$ such that for large a,

$$m_a \geq c\, a^{-1/2} e^{a^2T + a}. \tag{9}$$

Now let us consider the right-hand side.

$$M_a^2 = \int_0^T \int_0^1 \left(u_a^2 + (\partial_x u_a)^2 + (\partial_t u_a)^2 + (\partial_{xt}^2 u_a)^2 + (\partial_{xx}^2 u_a)^2\right)\, dxdt.$$

There exists a constant C such that for large a,

$$M_a^2 \leq C a^6 \int_0^T \int_0^1 e^{2ax+2a^2t}\, dxdt = C a^6\, \frac{e^{2a^2T} - 1}{2a^2} \frac{e^{2a} - 1}{2a}.$$

Hence there exists some constant $C > 0$ such that for large a,

$$M_a \leq C\, a^{3/2} e^{a^2T + a}. \tag{10}$$

On the other hand, we have

$$M_a^2 \geq \int_0^T \int_X^1 \left(v_a^2 + (\partial_x v_a)^2 + (\partial_t v_a)^2 + (\partial_{xt}^2 v_a)^2 + (\partial_{xx}^2 v_a)^2\right)\, dxdt.$$

We conclude as previously that there exists a constant c such that for large a,

$$M_a \geq c\, a^{3/2} e^{a^2T + a}. \tag{11}$$

Lastly, there exists some constant C such that

$$\delta_a^2 \leq C \int_0^T \int_0^X \left(v_a^2 + (\partial_x v_a)^2\right)\, dxdt.$$

We conclude that there exists some constant C such that for large a,

$$\delta_a \leq C a^{-1/2} e^{a^2T + aX}. \tag{12}$$

Plugging the estimates (9), (10), (11) and (12) in estimate (8) and using the fact that f in non-decreasing, we obtain that there exist some constants c, C such that for large a,

$$f(c\, a^2 e^{a(1-X)}) \leq C\, a^2.$$

We complete the proof by setting $A = c\, a^2 e^{a(1-X)}$.

Now let us address the 2D case, for which we have supposed that $\Omega = B(O,2) \setminus \overline{B(O,1)}$. Let us consider for all $p \in \mathbb{N}$, $p \neq 0$, the smooth functions defined on $\overline{\Omega} \times [0,T]$ by

$$u_p(x,t) = \chi(r) v_p(x,t), \quad v_p(x,t) = r^p \sin(p\,\theta),$$

where (r,θ) are the polar coordinates and χ is a smooth function of r on the segment $[1,2]$ such that $\chi(1) = \chi'(1) = 0$, $\chi(r) \in [0,1]$ for all $r \in [1,2]$ and $\chi(r) = 1$ for all $r \in [R,2]$ for some $R \in (1,2)$. Note that the functions u_p and v_p are independent of time t and that $u_p = 0$ and $\partial_r u_p = 0$ on the inner boundary $C(O,1)$, so that we have for all p

$$m_p \leq \frac{M_p}{f(M_p/\delta_p)}, \tag{13}$$

with

$$m_p = ||u_p||_{H^1(\Omega)}, \quad M_p = ||u_p||_{H^2(\Omega)}, \quad \delta_p = ||\Delta u_p||_{L^2(\Omega)}.$$

By repeating the same computations as for the 1D case, we prove that there exist two constants $c, C > 0$ such that for large p,

$$m_p \geq c\,p^{1/2}\,2^p, \quad c\,p^{3/2}\,2^p \leq M_p \leq C\,p^{3/2}\,2^p, \quad \delta_p \leq C\,p^{1/2} R^p. \tag{14}$$

As an example of such computation, the minoration of m_p is obtained as follows:

$$\begin{aligned} m_p^2 &= \int_1^2 \int_0^{2\pi} \left(u_p^2 + (\partial_r u_p)^2 + \frac{1}{r^2} (\partial_\theta u_p)^2 \right) r\, dr d\theta \\ &\geq \int_R^2 \int_0^{2\pi} \left(v_p^2 + (\partial_r v_p)^2 + \frac{1}{r^2} (\partial_\theta v_p)^2 \right) r\, dr d\theta \\ &= \int_R^2 \int_0^{2\pi} \left(r^{2p+1} \sin^2(p\,\theta) + p^2 r^{2p-1} \right) dr d\theta \\ &= \frac{\pi}{2p+2} (2^{2p+2} - R^{2p+2}) + \pi p\, (2^{2p} - R^{2p}), \end{aligned}$$

which implies the result. To obtain the majoration of δ_p it is useful to remark that $\Delta v_p = 0$ for all p. By plugging the estimates (14) in (13), we conclude that there exist some constants $c, C > 0$ such that for large p,

$$f(c\,p\,(2/R)^p) \leq C\,p,$$

and we conclude as in the 1D case. □

Remark 6. We hence prove that for $d = 2$, the estimate for $s = 1 - \varepsilon$ of our Main Theorem is quasi-optimal, in the sense that the result given by Theorem 5.1 contradicts the validity of the estimate for exponent $s > 1$, but not for $s = 1$. We can probably adapt the example for $d = 2$ to prove such quasi-optimality for any dimension $d > 2$.

6. **An application to the identification of initial condition.** In this section we come back to the general case of dimension d with Assumption 1.1. As an application of the Main Theorem, let us show that the initial condition, namely $u_0 := u|_{S_0}$ for $S_0 = \Omega \times \{0\}$, can be estimated from the data (f, g_0, g_1) in problem (1). More precisely, we have the following result.

Corollary 1. *For all $s \in (0,1)$, there exists a constant $C > 0$ such that for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$,*

$$||u_0||_{L^2(\Omega)} \leq C\, M(u) \left(\frac{1}{\log^s \left(2 + \frac{M(u)}{\delta(u)}\right)} + \frac{\delta(u)}{M(u)} \right)^{\frac{3}{4}},$$

where

$$\delta(u) = ||Pu||_{L^2(Q)} + ||u||_{H^1(\Sigma_0)} + ||\partial_n u||_{L^2(\Sigma_0)},$$
$$M(u) = ||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}.$$

Proof. Given some $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, it is useful to introduce for $(x,t) \in Q$ the vector field $\boldsymbol{U}(x,t) \in \mathbb{R}^{d+1}$ defined by

$$\boldsymbol{U} = (\nabla u, -u) = (\partial_{x_i} u, -u), \quad i = 1, \cdots, d.$$

Clearly, $\mathrm{div}_{d+1}\boldsymbol{U} = \Delta u - \partial_t u = -Pu$, so that by defining

$$H_{\mathrm{div}(Q)} = \{\boldsymbol{U} \in (L^2(Q))^{d+1},\ \mathrm{div}_{d+1}\boldsymbol{U} \in L^2(Q)\},$$

we have $\boldsymbol{U} \in H_{\mathrm{div}(Q)}$ and

$$||\boldsymbol{U}||^2_{H_{\mathrm{div}(Q)}} = ||u||^2_{L^2(0,T;H^1(\Omega))} + ||Pu||^2_{L^2(Q)}.$$

From a standard trace result in the Lipschitz domain Q, there exists a constant $C > 0$ such that for all $\boldsymbol{U} \in H_{\mathrm{div}(Q)}$,

$$||\boldsymbol{U} \cdot n_{d+1}||_{H^{-1/2}(\partial Q)} \leq C\, ||\boldsymbol{U}||_{H_{\mathrm{div}(Q)}},$$

where n_{d+1} is the outward unit normal to the domain Q. This implies in particular that $u_0 = u|_{S_0} \in H^{-1/2}(\Omega)$ and there exists a constant $C > 0$ such that for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$,

$$||u_0||_{H^{-1/2}(\Omega)} \leq C\, (||u||_{L^2(0,T;H^1(\Omega))} + ||Pu||_{L^2(Q)}),$$

where $H^{-1/2}(\Omega)$ is the space of restrictions to Ω of distributions in $H^{-1/2}(\mathbb{R}^d)$, that is by using the Main Theorem,

$$||u_0||_{H^{-1/2}(\Omega)} \leq C \left(\frac{M(u)}{\log^s \left(2 + \frac{M(u)}{\delta(u)}\right)} + \delta(u) \right). \tag{15}$$

We need an additional estimate for u_0 in a stronger norm. That $u \in H^1(0,T;H^1(\Omega)) \subset C^0([0,T];H^1(\Omega))$ immediately implies that $u_0 \in H^1(\Omega)$. But furthermore we can prove by using the second assumption $u \in L^2(0,T;H^2(\Omega))$ that $u_0 \in H^{3/2}(\Omega)$. Indeed we remark that $\boldsymbol{U} \in (H^1(Q))^{d+1}$ and

$$||\boldsymbol{U}||^2_{(H^1(Q))^{d+1}} = ||u||^2_{H^1(0,T;H^1(\Omega))} + ||u||^2_{L^2(0,T;H^2(\Omega))}.$$

From a second standard trace result in the domain Q, we have for all $\boldsymbol{U} \in (H^1(Q))^{d+1}$,

$$||\boldsymbol{U}||_{(H^{1/2}(\partial Q))^{d+1}} \leq C\, ||\boldsymbol{U}||_{(H^1(Q))^{d+1}},$$

which implies in particular, given the definition of $\boldsymbol{U}$, that the trace on S_0 of the partial derivatives $\partial_{x_i} u$, $i = 1, \cdots, d$, belongs to $H^{1/2}(\Omega)$. By density of $C^\infty(\overline{Q})$ in $H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, we clearly have

$$(\partial_{x_i} u)|_{S_0} = \partial_{x_i}(u|_{S_0}), \quad i = 1, \cdots, d,$$

that is $\partial_{x_i} u_0 \in H^{1/2}(\Omega)$. We finally end up with $u_0 \in H^{3/2}(\Omega)$ and there exists a constant $C > 0$ such that for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$,

$$||u_0||_{H^{3/2}(\Omega)} \leq C\,(||u||_{H^1(0,T;H^1(\Omega))} + ||u||_{L^2(0,T;H^2(\Omega))}),$$

that is

$$||u_0||_{H^{3/2}(\Omega)} \leq C\,M(u). \tag{16}$$

By an interpolation result of [18] (see Remark 12.6 applied to $s_1 = 1/2$, $s_2 = 3/2$ and $\theta = 1/4$), we have

$$\left[H^{1/2}_{00}(\Omega), (H^{3/2}(\Omega))'\right]_{1/4} = L^2(\Omega),$$

where $H^{1/2}_{00}(\Omega)$ is the subspace of functions in $H^{1/2}(\Omega)$ such that their extension by 0 to the whole space $\mathbb{R}^d$ belongs to $H^{1/2}(\mathbb{R}^d)$ and $(H^{3/2}(\Omega))'$ is the dual space of $H^{3/2}(\Omega)$. Since the dual space of $H^{1/2}_{00}(\Omega)$ coincides with space $H^{-1/2}(\Omega)$, by using Theorem 6.2 of [18] we obtain

$$\left[H^{3/2}(\Omega), H^{-1/2}(\Omega)\right]_{3/4} = L^2(\Omega),$$

as well as the interpolation inequality: there exists a constant $C > 0$ such that for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$,

$$||u_0||_{L^2(\Omega)} \leq C\,||u_0||^{1/4}_{H^{3/2}(\Omega)}||u_0||^{3/4}_{H^{-1/2}(\Omega)}.$$

The conclusion follows from the estimates (15) and (16). □

7. **An application to quasi-reversibility.** Again we consider the general case of dimension d with Assumption 1.1. The method of quasi-reversibility, first introduced in [15], is a regularization method that enables us to approximate the solution to the ill-posed problem (1) from the data (f, g_0, g_1) when such a solution exists (it is then unique). Let us assume that $f \in L^2(Q)$, $g_0 \in H^{3/2}(\Sigma_0)$ and $g_1 \in H^{1/2}(\Sigma_0)$, while $u \in H^2(Q)$ satisfies

$$\begin{cases} Pu = f & \text{in} \quad Q \\ u = g_0 & \text{on} \quad \Sigma_0 \\ \partial_n u = g_1 & \text{on} \quad \Sigma_0 \end{cases} \tag{17}$$

with $Pu = \partial_t u - \Delta u$. Let us introduce the subspace V of functions $v \in H^2(Q)$ such that $v|_{\Sigma_0} = 0$ and $\partial_n v|_{\Sigma_0} = 0$. For $\alpha > 0$, the quasi-reversibility solution u_α associated with any data $(\tilde{f}, \tilde{g}_0, \tilde{g}_1) \in L^2(Q) \times H^{3/2}(\Sigma_0) \times H^{1/2}(\Sigma_0)$ is the unique solution to the weak formulation: find $u_\alpha \in H^2(Q)$ such that

$$\begin{cases} (Pu_\alpha, Pv)_{L^2(Q)} + \alpha(u_\alpha, v)_{H^2(Q)} = (\tilde{f}, Pv)_{L^2(Q)} & \text{for all} \quad v \in V \\ u_\alpha = \tilde{g}_0 & \text{on} \quad \Sigma_0 \\ \partial_n u_\alpha = \tilde{g}_1 & \text{on} \quad \Sigma_0. \end{cases} \tag{18}$$

The unique solvability of problem (18) relies on Lax-Milgram's Lemma once we have introduced some $U \in H^2(Q)$ such that $U|_{\Sigma_0} = \tilde{g}_0$ and $\partial_n U|_{\Sigma_0} = \tilde{g}_1$, which is possible in virtue of an extension theorem of [18], as well as the auxiliary unknown $\hat{u}_\alpha = u_\alpha - U$. Furthermore, if the data $(\tilde{f}, \tilde{g}_0, \tilde{g}_1)$ coincides with the exact data (f, g_0, g_1), it is easy to prove the estimates

$$||u_\alpha - u||_{H^2(Q)} \leq ||u||_{H^2(Q)}, \quad ||P(u_\alpha - u)||_{L^2(Q)} \leq \sqrt{\alpha}\,||u||_{H^2(Q)}, \quad \forall \alpha > 0, \tag{19}$$

as well as the fact that $u_\alpha \to u$ in $H^2(Q)$ when $\alpha \to 0$ (see for example [15]). A more difficult question is to specify the convergence rate. The first step in this direction was done by M.V. Klibanov (see for example [14]), more precisely some Hölder-type convergence rates were obtained in the truncated domain $\Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$. However it is desirable to obtain a convergence rate in the whole domain $Q = \Omega \times (0,T)$ as it was done for the Laplacian in [4, 5]. Our previous results enable us to derive a logarithmic convergence rate in the whole domain.

Corollary 2. *For all $s \in (0,1)$, there exists a constant $C > 0$ such that for all $u \in H^2(Q)$ satisfying (17) for data (f, g_0, g_1) and small $\alpha > 0$,*

$$||u_\alpha - u||_{L^2(0,T;H^1(\Omega))} \leq C \frac{||u||_{H^2(Q)}}{\log^s(1/\alpha)},$$

where u_α is the solution to problem (18) associated with the data (f, g_0, g_1).

Proof. To obtain the result, we simply use the estimates (19), the continuous embedding $H^2(Q) \subset H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and the Main Theorem (in the form given in Remark 3) applied to the function $(u_\alpha - u)$ with $M = ||u||_{H^2(Q)}$ and $\delta = \sqrt{\alpha}\, ||u||_{H^2(Q)}$. □

Remark 7. In the above corollary, the solutions u_α, u seem surprisingly smooth. There are two reasons. Firstly, the well-posedness of the weak formulation of quasi-reversibility (18) requires a lifting in Q of the boundary data $(\tilde{g}_0, \tilde{g}_1)$ on Σ_0, which have to be sufficiently smooth. Secondly, the Main Theorem requires the solutions to be in $H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$. For example, following the notations of [19], it would be interesting and more natural to consider the less regular case when $\tilde{f} \in L^2(Q)$, $\tilde{g}_0 \in H^{3/2,3/4}(\Sigma_0)$, $\tilde{g}_1 \in H^{1/2,1/4}(\Sigma_0)$ and solutions u_α, u in $H^{2,1}(Q)$. Those boundary data $(\tilde{g}_0, \tilde{g}_1)$ can be lifted in $H^{2,1}(Q)$ but unfortunately, the space $H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ does not include $H^{2,1}(Q)$.

8. **About Assumption 1.1.** As can be seen in the proof of our Main Theorem, it was possible in a single step to estimate the solution in the subdomain $\Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$ for small $\varepsilon > 0$ with constants that explicitly depend on ε because of the geometric assumption 1.1. Such assumption implies in particular that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, where Γ_1 is the complementary part of Γ_0 in $\partial\Omega$ and then enabled us to use a smooth spatial function η in the Carleman weight that coincides near the outer or inner boundary with the distance function to the boundary. Such general strategy is no more applicable without Assumption 1.1. However, it is a natural question whether our Main Theorem can be extended to the case when $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$.

The result given in Remark 5 for the Laplacian in Ω is obtained in [20, 4] without Assumption 1.1, by using a more complicated strategy. More precisely, a three steps strategy is used: the smallness of the function is propagated firstly from Γ_0 to the interior of Ω, secondly inside Ω from an open domain to another and thirdly from inside Ω up to the boundary Γ_1. Such strategy is probably feasible in the case of the heat equation, but the main difficulty is, for each of these three steps, to estimate the constants uniformly with respect to ε. This is a challenging future work.

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E-mail address: laurent.bourgeois@ensta.fr