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Abstract: We address in this work the uniqueness issue in the classical Robin inverse problem with the Laplace equation on a Dini-smooth domain $\Omega \subset \mathbb{R}^2$, with L^∞ Robin coefficient and L^2 Neumann data. We prove uniqueness of the Robin coefficient on a subpart of the boundary, given Cauchy data on the complementary part.

Key-words: Robin inverse problem, Dini-smooth domain, complex analysis, holomorphic functions, Hardy space H^2, unique continuation.

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Résultats d'unicité pour le problème inverse de Robin 2D avec coefficient borné

Résumé : Ce travail concerne la question de l'unicité pour le problème inverse de Robin avec équation de Laplace dans un domaine $\Omega \subset \mathbb{R}^2$ régulier au sens de Dini, le coefficient de Robin étant dans L^∞ et la donnée de Neumann dans L^2. Nous prouvons l'unicité du coefficient de Robin sur une partie du bord connaissant les données de Cauchy sur la partie complémentaire.

Mots-clés : Problème inverse de Robin, domaine Dini-régulier, analyse complexe, fonctions holomorphes, espace de Hardy H^2, prolongement unique.

1 Introduction

This study deals with uniqueness issues for the classical Robin inverse boundary value problem. Mathematically speaking, the inverse Robin problem for an elliptic partial differential equation on a domain consists in finding the ratio between the normal derivative and the trace of the solution (the so-called Robin coefficient) on a subset of the boundary, granted the Cauchy data (*i.e.* the normal derivative and the trace of the solution) on the complementary subset. Here, we deal with L^∞ Robin coefficients and L^2 Neumann data, for the Laplace equation on Dini-smooth domains $\Omega \subset \mathbb{R}^2$.
The main results (and the present introduction) are part of our work [2], where we addressed the same uniqueness problem in a more general situation. There, on the one hand, we consider a Lipschitz-smooth domain Ω instead of a Dini-smooth one. On the second hand, we consider isotropic conductivity equations instead of the Laplace equation, with a conductivity chosen in the Sobolev class $W^{1,r}(\Omega)$ where $r > 2$. These more general assumptions, which concerns both the regularity of the domain and the equation that governs the physics of the problem, required us to introduce in [2] rather sophisticated tools from complex and harmonic analysis. Our objective in the present research report is to describe and establish our uniqueness results in a less technical mathematical framework, for the simpler case of Laplace equation in a Dini-smooth domain.
The Robin inverse problem arises for example when considering non-destructive testing of corrosion in an electrostatic conductor. In this case, data consist of surface measurements of both the current and the voltage on some (accessible) part of the boundary of the conductor, while the complementary (inaccessible) part of the boundary is subject to corrosion. Non-destructive testing consists in quantifying corrosion from the data. Robin boundary condition can be regarded as a simple model for corrosion [10]. Indeed, as was proved in [5], such boundary conditions arise when considering a thin oscillating coating surrounding a homogeneous background medium such that the thickness of the layer and the wavelength of the oscillations tend simultaneously to 0. A mathematical framework for corrosion detection can then be described as follows. We consider a Laplace equation in an open domain Ω, the boundary of which is divided into two parts. The first part Γ is characterized by a homogeneous Robin condition with functional coefficient λ. A non vanishing flux is imposed on the second part Γ_0 of the boundary. This provides us with a well-posed forward problem, that is, there uniquely exists a solution in Ω meeting the prescribed boundary conditions. The inverse problem consists in recovering the unknown Robin coefficient λ on Γ from measurements of the trace of the solution on Γ_0. Further motivation to solve the Robin problem are indicated in [12] and its bibliography.
A basic question is uniqueness: is the coefficient λ on Γ uniquely defined by the

available Cauchy data on Γ_0 as soon as the latter has positive measure? In other words, can we find two different Robin coefficients that produce the same measurements? The answer naturally depends on the smoothness assumed for λ.
On smooth domains, uniqueness of the inverse Robin problem for (piecewise) continuous λ has been known for decades to hold in all dimensions. The proof is for example given in [10], and in [7] for the Helmholtz equation. It relies on a strong unique continuation property (Holmgren's theorem), *i.e.* on the fact that a harmonic function in Ω, the trace and normal derivative of which both vanish on a non-empty open subset of the boundary $\partial\Omega$, vanishes identically.
This argument no longer works for functions λ that are merely bounded. In this case we meet the following weaker unique continuation problem: *does a harmonic function, the trace and normal derivative of which both vanish on a subset of $\partial\Omega$ with positive measure, vanish identically?* A famous counterexample in [3] shows that such a unique continuation result is false in dimension 3 and higher. In dimension 2, a proof that such a unique continuation property holds for the Laplace equation can be found in [1] when the solution is assumed to be C^1 up to the boundary and Ω is the unit disk.
In this work, we prove more generally that this unique continuation result still holds in a simply connected domain $\Omega \subset \mathbb{R}^2$ bounded by a Dini-smooth curve, for a harmonic function u that belongs to the Sobolev class $W^{1,2}(\Omega)$ and that admits a L^2 normal derivative on the boundary $\partial\Omega$. This enables us to conclude to uniqueness in the inverse Robin problem. Our present proofs rely on classical tools from complex analysis, more specifically conformal mappings and a fundamental uniqueness property of holomorphic functions in Hardy classes from their values on boundary subsets of positive measure, as well as on a Rolle-type theorem for $W^{1,2}$ Sobolev functions on the real line.
Our uniqueness result for the Robin inverse problem generalizes that of [6] established in smoother cases and under the restriction that the imposed flux is non negative. The proof therein is based on positivity and monotonicity arguments and does not use complex analysis.
We finally point out that the counterexample of [3] is turned in [2] into a counterexample to uniqueness in the Robin problem in dimension 3.
The research report is organized as follows. The inverse Robin problem, in particular the statement of our uniqueness result, is presented in Section 2. Proofs of the results given in Section 2 are provided in Section 3. Section 4 contains a discussion about the more general results and techniques from [2].

2 The inverse Robin problem

2.1 Preliminaries

2.1.1 Notation

For an open domain $\Omega \subset \mathbb{R}^2$ and a boundary subset $\Gamma \subseteq \partial\Omega$, we will make use of the classical Lebesgue spaces $L^p(\Gamma)$, for $p = 1, 2, \infty$, together with the Sobolev-Hilbert spaces $W^{1,2}(\Omega)$, $W^{1,2}(\partial\Omega)$ and $W^{1,2}(0,1)$ [4]. The space $C^1([0,1])$ will be simply denoted $C^1(0,1)$.
We put $\mathbb{D} \subset \mathbb{C}$ for the unit disk of the complex plane and $\mathbb{T} = \partial\mathbb{D}$ for the unit circle.
Throughout, $l(\cdot)$ denotes the 1-dimensional Lebesgue measure.
Let $L^\infty_+(\Gamma) := \{\lambda \in L^\infty(\Gamma),\ \lambda \geq 0$ a.e. on $\Gamma,\ \lambda \not\equiv 0\}$ (there exists a subset of Γ with positive Lebesgue measure on which λ does not vanish).
Partial derivatives will be written ∂_ν (normal derivative), or ∂_{x_i}, $i = 1, 2$, ∂_z, $\partial_{\bar{z}}$ (derivatives w.r.t. coordinates x_i, associated complex affix z, or its conjugate $\bar{z}$, respectively); when no confusion occurs, the first derivative may be indicated by a prime $'$.

2.1.2 Dini-smooth domains

Let us consider a continuous function $\rho : [0, 2\pi] \mapsto \mathbb{R}_+$, its modulus of continuity is defined by [8]: $\omega(\delta) = \sup\{|\rho(t_1) - \rho(t_2)|,\ \ t_1, t_2 \in [0, 2\pi],\ |t_1 - t_2| \leq \delta\}$. Since ρ is a uniformly continuous function, we have that $\omega(\delta) \to 0$ when $\delta \to 0$. The function ρ is said to be Dini-continuous if, in addition, the non negative function $t \mapsto \omega(t)/t$ is integrable in a vicinity of 0. In particular, a Hölder-continuous function with exponent $\alpha \in (0, 1]$ is Dini-continuous.
Then, a Jordan curve in $\mathbb{R}^2$ is said to be Dini-smooth if it admits a differentiable parametrization (on $[0, 2\pi]$), whose derivative is a non vanishing Dini-continuous function [15, Sec. 3.3].
A simply connected domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is a Dini-smooth Jordan curve is called a Dini-smooth domain.
Observe that a Dini-smooth domain is of class C^1, whence also Lipschitz-smooth.

2.1.3 Geometrical assumptions

Throughout, we consider a bounded simply connected Dini-smooth domain $\Omega \subset \mathbb{R}^2$. It's boundary $\partial\Omega$ is partitioned into two non-empty subsets Γ and Γ_0: $\partial\Omega = \Gamma \cup \Gamma_0$ and $\Gamma \cap \Gamma_0 = \emptyset$, such that $l(\Gamma) > 0$, $l(\Gamma_0) > 0$.

2.2 Forward problem

Let $g \in L^2(\Gamma_0)$ such that $g \not\equiv 0$ and $\lambda \in L^\infty_+(\Gamma)$.
The forward problem we look at consists in finding $u \in W^{1,2}(\Omega)$ solution to

$$\left\{ \begin{array}{l} \Delta u = 0 \text{ in } \Omega \\ \partial_\nu u = g \text{ on } \Gamma_0 \\ \partial_\nu u + \lambda u = 0 \text{ on } \Gamma, \end{array} \right. \tag{1}$$

where ν is the outward unit normal of Ω. Problem (1) admits the following weak formulation: find $u \in W^{1,2}(\Omega)$ such that for all $v \in W^{1,2}(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma \lambda uv \, d\tau = \int_{\Gamma_0} gv \, d\tau. \tag{2}$$

Well-posedness of problem (2) follows from Lemma 2.1 below and Lax-Milgram's theorem [4, Cor. V.8], which ensures existence and uniqueness of a solution $u \in W^{1,2}(\Omega)$. Indeed, whenever $\lambda \in L^\infty_+(\Gamma)$ we have the following result, which implies a Poincaré-Friedrichs type inequality together with the coercivity of the bilinear form in the left-hand side of (2).

Lemma 2.1. *On $W^{1,2}(\Omega)$, the norm whose square is defined by*

$$\int_\Omega |\nabla u|^2 \, dx + \int_\Gamma \lambda \, u^2 \, d\tau$$

is equivalent to the standard $\|\cdot\|_{W^{1,2}(\Omega)}$ one.

Proof. Let us establish that there exist two constants $c, C > 0$ such that

$$c \, \|u\|^2_{W^{1,2}(\Omega)} \leq \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma \lambda \, u^2 \, d\tau \leq C \, \|u\|^2_{W^{1,2}(\Omega)} \, , \quad \forall u \in W^{1,2}(\Omega).$$

The right inequality comes from the hypothesis that $\lambda \in L^\infty_+(\Gamma)$, and from the standard trace inequality on $\partial\Omega$ for $W^{1,2}(\Omega)$ functions. Indeed, the trace operator $W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is continuous for Lipschitz-smooth domains Ω [14]. Let us prove the left inequality. Assume to the contrary that there exists a sequence (u_n) in $W^{1,2}(\Omega)$ such that

$$\|u_n\|_{W^{1,2}(\Omega)} = 1, \quad \int_\Omega |\nabla u_n|^2 \, dx + \int_\Gamma \lambda \, u_n^2 \, d\tau \leq \frac{1}{n}, \quad \forall n \geq 1. \tag{3}$$

Since Ω is a Lipschitz-smooth domain, the compact injection $W^{1,2}(\Omega) \subset L^2(\Omega)$ implies that we can extract a subsequence from u_n, still denoted u_n, such that

$u_n \to u$ in $L^2(\Omega)$, as $n \to \infty$, for some $u \in L^2(\Omega)$. Besides, the inequality in (3) then implies that

$$\|\nabla u_n\|^2_{L^2(\Omega)} \leq \frac{1}{n},$$

whence (u_n) satisfies the Cauchy criterium in $W^{1,2}(\Omega)$. Thus $u_n \to u$ in $W^{1,2}(\Omega)$ and

$$\|\nabla u_n\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

Hence, a.e. in Ω, we have $\nabla u = 0$, and u is equal to a constant C_u. Further, as $n \to \infty$,

$$\int_\Gamma \lambda\, u_n^2\, d\tau \to 0 = \int_\Gamma \lambda\, u^2\, d\tau = C_u^2 \int_\Gamma \lambda\, d\tau.$$

Since λ does not identically vanish, we necessarily have $C_u = 0$, which contradicts the fact that $\|u\|_{W^{1,2}(\Omega)} = 1$ from the equality in (3). □

2.3 Inverse problem, uniqueness results

The associated Robin inverse problem consists in finding some unknown impedance λ in $L^\infty_+(\Gamma)$ from available measurements (Dirichlet boundary data) $u_{|\Gamma_0}$ in $L^2(\Gamma_0)$ of the solution u to (1) on Γ_0.
The following uniqueness theorems are the main results of this work, for domains $\Omega \subset \mathbb{R}^2$ that satisfy the assumptions of Section 2.1.3.

Theorem 2.2. *Let $g \in L^2(\Gamma_0)$, $g \not\equiv 0$, and $\lambda_1, \lambda_2 \in L^\infty_+(\Gamma)$ such that the corresponding solutions $u_1, u_2 \in W^{1,2}(\Omega)$ to problem (1) satisfy $u_{1|\Gamma_0} = u_{2|\Gamma_0}$. Then $\lambda_1 = \lambda_2$.*

It is a consequence of the following result, whose proof is given in Section 3.

Theorem 2.3. *Let $u \in W^{1,2}(\Omega)$ be harmonic in Ω and such that $\partial_\nu u \in L^2(\partial\Omega)$. If both u and $\partial_\nu u$ vanish on a subset $\gamma \subset \partial\Omega$ of positive Lebesgue measure, then $u \equiv 0$ in Ω.*

Proof. (of Theorem 2.2) The solutions u_1 and u_2 are both harmonic in Ω and have normal derivatives in $L^2(\partial\Omega)$. By assumption, u_1 and u_2 have the same Cauchy data on $\Gamma_0 \subset \partial\Omega$ with $l(\Gamma_0) > 0$. Theorem 2.3 then implies that $u_1 \equiv u_2$ in Ω, so that $u_{1|\Gamma}$ and $u_{2|\Gamma}$ coincide, as well as $\partial_\nu u_{1|\Gamma}$ and $\partial_\nu u_{2|\Gamma}$. Whence $(\lambda_1 - \lambda_2)\, u_1 = 0$ on Γ, from Robin boundary condition.
Assume that $\lambda_1 \neq \lambda_2$ a.e. on Γ: there exists a subset $\gamma \subset \Gamma$, $l(\gamma) > 0$, such that $\lambda_1 - \lambda_2 \neq 0$ on γ. Then u_1 vanishes on γ and so does $\partial_\nu u_1$, since $\partial_\nu u_1 = -\lambda_1\, u_1$. From Theorem 2.3 again, this implies $u_1 \equiv 0$ in Ω, thus $\partial_\nu u_1 = 0$ a.e. on $\partial\Omega$ which contradicts the assumption $g \not\equiv 0$ in Γ_0. □

3 Proof of Theorem 2.3

3.1 Holomorphic functions, conformal mapping

For any harmonic function u in the simply connected domain Ω, let us introduce the corresponding complex valued function f of the complex variable $z = x_1 + ix_2 \in \mathbb{C}$, associated to $x = (x_1, x_2) \in \mathbb{R}^2$, such that $u = \mathrm{Re}\, f$ and defined by [16, Ch. 11]:

$$f(z) = u(x) + iv(x)\,,$$

where v is the conjugate harmonic function associated to u in Ω, defined up to an additive constant by the Cauchy-Riemann equations:

$$\partial_{x_1} v = -\partial_{x_2} u\,,\ \partial_{x_2} v = \partial_{x_1} u\,.$$

Hence, the function f is holomorphic in Ω [16, Ch. 11], since it satisfies:

$$\partial_{\bar{z}} f = \frac{1}{2}\,(\partial_{x_1} f + i\partial_{x_2} f) = 0\,,$$

and its holomorphic derivative is given there by:

$$f' = \partial_z f = \frac{1}{2}\,(\partial_{x_1} f - i\partial_{x_2} f) = \partial_{x_1} u - i\partial_{x_2} u = 2\,\partial_z u. \tag{4}$$

Besides, because $\partial\Omega$ is a Jordan curve, the Riemann mapping theorem [16, Thm 14.19] is to the effect that there exists a conformal mapping φ from Ω onto the unit disk $\mathbb{D} \subset \mathbb{C}$, which extends to a homeomorphism from $\overline{\Omega}$ onto $\overline{\mathbb{D}}$. Recall that a conformal mapping is a holomorphic function whose derivative (w.r.t. z) does not vanish. Since $\partial\Omega$ is further assumed to be a Dini-smooth curve, it also holds from [15, Thm 3.5] that φ admits a continuous derivative that does not vanish on $\overline{\Omega}$. This is naturally true for the inverse map $\psi = \varphi^{-1} : \overline{\mathbb{D}} \to \overline{\Omega}$ as well. Hence, ψ is continuously differentiable in $\overline{\mathbb{D}}$ and ψ' does not vanish in $\overline{\mathbb{D}}$, which implies that:

$$\exists \text{ constants } c, C > 0 \text{ such that } \forall z \in \overline{\mathbb{D}}\,,\ c \leq |\psi'(z)| \leq C\,. \tag{5}$$

Composition by such a conformal mapping associates the holomorphic function f on Ω to the holomorphic function $F = f \circ \psi$ on $\mathbb{D}$ (holomorphy and harmonicity are preserved by composition with conformal maps).

3.2 Hardy space $H^2(\mathbb{D})$

Recall (see [8] or [16, Ch. 17]) that a holomorphic function g in $\mathbb{D}$ with complex valued Fourier coefficients (g_n), $n \geq 0$, belongs to H^2 if and only if $(g_n) \in l^2(\mathbb{N})$:

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \in H^2 \Leftrightarrow \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left|g(r\, e^{i\theta})\right|^2\, d\theta = \sum_{n=0}^{\infty} |g_n|^2 < +\infty.$$

In particular, from [16, Thm 17.10], $g \in H^2$ admits a radial limit up to the boundary, $g_{|\mathbb{T}} \in L^2(\mathbb{T})$, such that for a.e. $e^{i\theta} \in \mathbb{T}$:

$$\lim_{r\to 1} \left|g(re^{i\theta}) - g_{|\mathbb{T}}(e^{i\theta})\right| = 0, \text{ and } \lim_{r\to 1} \int_0^{2\pi} \left|g(re^{i\theta}) - g_{|\mathbb{T}}(e^{i\theta})\right|^2 d\theta = 0\,.$$

Alternatively, boundary values (radial or non-tangential limits) on $\mathbb{T}$ of H^2 functions form the subspace of $L^2(\mathbb{T})$ whose Fourier coefficients of negative indices vanish.
The unique continuation result which is fundamental for the proof of Theorem 2.3 is the following result [8, 11, 16]:

Theorem 3.1. *If G belongs to the Hardy space H^2 and it's radial limit $G_{|\mathbb{T}}$ (defined in $L^2(\mathbb{T})$) vanishes on a subset of $\mathbb{T}$ with positive measure, then $G \equiv 0$ in $\overline{\mathbb{D}}$.*

Since it is crucial here, we give a sketch of its proof following [16, Thm 17.18].

Proof. It is enough to prove that if G does not vanish identically in $\mathbb{D}$ then $\log|G_{|\mathbb{T}}| \in L^1(\mathbb{T})$. We first remark that $H^2 \subset N$, where N is the set of holomorphic functions G such that

$$\sup_{0\le r<1} m_0(r) < +\infty, \quad m_0(r) := \int_0^{2\pi} \log^+ \left|G(r\, e^{i\theta})\right| \, d\theta,$$

where $\cdot^+$ is meant for the non negative part. Since the real-valued function $z \mapsto \log^+|G(z)|$ is continuous and subharmonic in $\mathbb{D}$ (G is not identically 0), the function $r \mapsto m_0(r)$ is non decreasing, so that the supremum over $r < 1$ in the previous definition of N coincides with the limit when $r \to 1^-$.
The second step consists in pointing out that if B denotes the Blaschke product formed by the zeroes of the function G [16, Thm 15.21], then since $G \in H^2$, the function $\tilde{G} = G/B$ belongs to H^2 as well and satisfies $\tilde{G}_{|\mathbb{T}} = G_{|\mathbb{T}}$ *a.e.* on $\mathbb{T}$, so that it suffices to prove the theorem for $\tilde{G}$ instead of G.
In the last step, G is now a function of $H^2 \subset N$ which does not vanish in $\mathbb{D}$, so that the real-valued function $z \mapsto \log|G(z)|$ is harmonic in $\mathbb{D}$. We may set $G(0) = 1$ without loss of generality. As a consequence of the mean value characterization of harmonic functions, we have for $r < 1$

$$\log|G(0)| = 0 = \int_0^{2\pi} \log\left|G(r\, e^{i\theta})\right| \, d\theta,$$

and then, by using the fact that $\log = \log^+ - \log^-$,

$$\int_0^{2\pi} \log^- \left|G(r\, e^{i\theta})\right| \, d\theta = \int_0^{2\pi} \log^+ \left|G(r\, e^{i\theta})\right| \, d\theta \le \sup_{0\le r<1} m_0(r) < +\infty.$$

From Fatou's lemma, we conclude that both functions $\log^+|G_{|\mathbb{T}}|$ and $\log^-|G_{|\mathbb{T}}|$ belong to $L^1(\mathbb{T})$, whence $\log|G_{|\mathbb{T}}|$ belongs to $L^1(\mathbb{T})$, which completes the proof. □

3.3 An extended Rolle-type theorem

We prove below that if a $W^{1,2}$ function defined on a segment of the real line $\mathbb{R}$ vanishes on a set B of positive measure, then its derivative (in the sense of distributions) also vanishes on a subset $B' \subset B$ of positive measure.

Theorem 3.2. *Let us consider a function $u \in W^{1,2}(0,1)$, such that $u = 0$ in a subset $B \in [0,1]$ with $l(B) > 0$. Then there exists $B' \subset B$ with $l(B') > 0$ such that $u' = 0$ in B'.*

Such a result is obvious when B is an open set. In the general case, it is a consequence of an extension of Lusin's theorem for $W^{1,2}$ functions (Lemma 3.3), and of the fact that isolated points of any subset of $\mathbb{R}$ form a countable set (Lemma 3.4), together with Rolle's theorem (for C^1 functions).
Lemma 3.3 is an intermediate value theorem which holds as a particular case of [17, Thm 3.10.5] (where the result is stated in $\mathbb{R}$ and from which the present statement in $(0,1)$ directly follows, using the extension theorem):

Lemma 3.3. *Let $u \in W^{1,2}(0,1)$ and $\varepsilon > 0$. There exists an open set $\mathcal{U} \subset (0,1)$ and a function $v \in C^1(0,1)$ such that $l(\mathcal{U}) < \varepsilon$ and $v(t) = u(t)$, $v'(t) = u'(t)$, $\forall t \in (0,1) \setminus \mathcal{U}$.*

Lemma 3.4 may be found in [9]. However we give a proof the sake of completeness.

Lemma 3.4. *Let B denote a subset of $[0,1]$. The subset I formed by the isolated points of B is countable.*

Proof. For $t \in I \setminus \{\sup I\}$, let $d(t) = \inf\{s - t,\ s > t, s \in I\}$, which is well defined. Since any point t is an isolated point of B, it is also an isolated point of I, whence $d(t) > 0$. But the intervals $(t, t + d(t))$, $t \in I \setminus \{\sup I\}$, are non-overlapping and all contained in $[0,1]$. Hence the family $\{d(t),\ t \in I \setminus \{\sup I\}\}$ is summable, its sum is bounded by 1 and its support is equal to the set $I \setminus \{\sup I\}$. As a consequence of a classical result on summable families, the support of such a family is countable. The set I is hence countable. □

We are now in a position to give a proof of Theorem 3.2.

Proof. (of Theorem 3.2) Consider a function u which satisfies the assumptions of the theorem. Choose $\varepsilon \in (0, l(B))$ and consider the associated open set $\mathcal{U}$ and function v from lemma 3.3. We then define $B_s = B \setminus \mathcal{U} = B \cap ([0,1] \setminus \mathcal{U})$. If we had $l(B_s) = 0$, then

$$l(B \cup ([0,1] \setminus \mathcal{U})) = l(B) + l([0,1] \setminus \mathcal{U}) = l(B) + 1 - l(\mathcal{U}) > 1,$$

because $l(\mathcal{U}) < \varepsilon < l(B)$ by assumptions, which is impossible since $B \subset [0,1]$. Thus $l(B_s) > 0$. The set B_s has the decomposition $B_s = I \cup A$, $I \cap A = \emptyset$, where I and A denote the sets of isolated and accumulation points of B_s, respectively. From Lemma 3.4, the set I is countable, whence $l(I) = 0$, and then $l(A) = l(B_s) > 0$.
Lastly, let us consider $t \in A$. There exists a non-stationary sequence (t_n) in the set B_s, $n \in \mathbb{N}$, such that $t_n \to t$. Without loss of generality we may assume that $t_n < t_{n+1}$, for all $n \in \mathbb{N}$. The sequence (t_n) satisfies $u(t_n) = v(t_n)$ for all $n \in \mathbb{N}$, so that, by applying Rolle's theorem to the function $v \in C^1(0,1)$, we get that for all n, there exists $s_n \in (t_n, t_{n+1})$ such that

$$0 = u(t_{n+1}) - u(t_n) = v(t_{n+1}) - v(t_n) = v'(s_n)(t_{n+1} - t_n).$$

This implies that $v'(s_n) = 0$ for all $n \in \mathbb{N}$, and by passing to the limit, $v'(t) = 0$, that is $u'(t) = 0$. We conclude that u' vanishes on the subset $A \subset B$ with $l(A) > 0$, which establishes the result with $B' := A$. □

3.4 Proof of Theorem 2.3, with a regularity result

Let us consider a function u as in the statement of Theorem 2.3.
Step 1. Denote $U = u \circ \psi = \mathrm{Re}\, F$, where ψ and F are defined as in Section 3.1.
Step 2. Now let us prove that $F' = (f \circ \psi)'$ belongs to the Hardy space $H^2 = H^2(\mathbb{D})$ of the unit disk defined in Section 3.2.
Since F is holomorphic on $\mathbb{D}$, U in harmonic in $\mathbb{D}$. Next, it follows from property (5) of ψ that $u \in W^{1,2}(\Omega)$ implies that $U \in W^{1,2}(\mathbb{D})$. Besides, some easy computations leads the to the following relationships between the normal (resp. tangential) derivative of U on $\mathbb{T}$ and the corresponding normal (resp. tangential) derivative of u on $\partial\Omega$: for all $\theta \in [0, 2\pi]$,

$$\partial_r U(e^{i\theta}) = |\psi'(e^{i\theta})|\, (\partial_\nu u \circ \psi)(e^{i\theta}), \quad \partial_\theta U(e^{i\theta}) = |\psi'(e^{i\theta})|\, (\partial_\tau u \circ \psi)(e^{i\theta}). \tag{6}$$

By using the first identity of (6) and the fact that the Lebesgue measure on $\partial\Omega$ is related to the Lebesgue measure on $\mathbb{T}$ by $d\tau = |\psi'(e^{i\theta})|\, d\theta$,

$$\int_{\partial\Omega} |\partial_\nu u|^2 \, d\tau = \int_0^{2\pi} |\psi'(e^{i\theta})| \, \left|(\partial_\nu u \circ \psi)(e^{i\theta})\right|^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{|\psi'(e^{i\theta})|} \left|\partial_r U(e^{i\theta})\right|^2 \, d\theta \geq \frac{1}{C} \int_0^{2\pi} \left|\partial_r U(e^{i\theta})\right|^2 \, d\theta.$$

We conclude that the normal derivative $\partial_r U_{|\mathbb{T}}$ belongs to $L^2(\mathbb{T})$.
A classical interpolation result for smooth domains [13] ensures that since $U \in W^{1,2}(\mathbb{D})$ is harmonic in $\mathbb{D}$ and has a normal derivative $\partial_r U_{|\mathbb{T}}$ in $L^2(\mathbb{T})$, then U

belongs to the Sobolev space $W^{3/2,2}(\mathbb{D})$, whence its trace $U_{|\mathbb{T}}$ on $\mathbb{T}$ belongs to $W^{1,2}(\mathbb{T})$ [4, Ch. IX]. Now, by using the second identity of (6), we obtain

$$\int_{\partial\Omega} |\partial_\tau u|^2 \, d\tau = \int_0^{2\pi} |\psi'(e^{i\theta})| \, \left|(\partial_\tau u \circ \psi)(e^{i\theta})\right|^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{|\psi'(e^{i\theta})|} \left|\partial_\theta U(e^{i\theta})\right|^2 \, d\theta \leq \frac{1}{c} \int_0^{2\pi} \left|\partial_\theta U(e^{i\theta})\right|^2 \, d\theta.$$

This proves that the trace $u_{|\partial\Omega}$ of u on $\partial\Omega$ satisfies $u_{|\partial\Omega} \in W^{1,2}(\partial\Omega)$, and the following regularity result.

Proposition 1. *Let $u \in W^{1,2}(\Omega)$ be harmonic in Ω and such that $\partial_\nu u \in L^2(\partial\Omega)$. Then, its trace $u_{|\partial\Omega}$ on $\partial\Omega$ belongs to $W^{1,2}(\partial\Omega)$.*

In particular, $U_{|\mathbb{T}} \in C(\mathbb{T})$ and the holomorphic function F admits the following Poisson representation in $\mathbb{D}$ [8, Sec. 4.4], [16, Thm 11.12]:

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} U_{|\mathbb{T}}(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta,$$

so that

$$F'(z) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{i\theta} \, U_{|\mathbb{T}}(e^{i\theta})}{(e^{i\theta} - z)^2} \, d\theta.$$

From the fact that, for $z \in \mathbb{D}$,

$$\frac{e^{i\theta}}{(e^{i\theta} - z)^2} = \sum_{n=0}^{+\infty} e^{-i(n+1)\theta} (n+1) z^n,$$

we obtain

$$F'(z) = \sum_{n=0}^{+\infty} a_n z^n,$$

where, for $n \in \mathbb{N}$,

$$a_n := (n+1) b_n, \quad b_n := \frac{1}{\pi} \int_0^{2\pi} U_{|\mathbb{T}}(e^{i\theta}) e^{-i(n+1)\theta} \, d\theta \, .$$

Hence, the b_n are the Fourier coefficients of the function $2e^{-i\theta} U_{|\mathbb{T}}(e^{i\theta})$ which belongs to $W^{1,2}(\mathbb{T})$, since $U_{|\mathbb{T}}$ does. Thus, $(a_n) \in l^2(\mathbb{N})$, and this implies that the holomorphic function F' belongs to the Hardy space H^2.

Step 3. In order to apply Theorem 3.2 from Section 3.3, we introduce the parametrization of $\partial\Omega$ given by:

$$t \in [0,1] \mapsto \Psi(t) = \psi(e^{2i\pi t}),$$

we get from Proposition 1 that the function

$$t \in [0,1] \mapsto \tilde{u}(t) = (u \circ \Psi)(t),$$

belongs to $W^{1,2}(0,1)$ and vanishes on $\Psi^{-1}(\gamma) = B \subset [0,1]$, by assumption. We have, for all $t \in [0,1]$,

$$\tilde{u}'(t) = (u \circ \Psi)'(t) = |\Psi'(t)| \, (\partial_\tau u \circ \Psi)(t).$$

From Theorem 3.2, there exists a subset $B' \subset B$ of positive measure, on which the derivative $\tilde{u}'$ of $\tilde{u}$ also vanishes. In turn, by using the fact that the function $\Psi'(t)$ does not vanish in $[0,1]$ (because $|\Psi'(t)| = 2\pi \, |\psi'(e^{2i\pi t})| \neq 0$) we conclude that the tangential derivative $\partial_\tau u$ vanishes on a subset $\gamma' = \Psi(B') \subset \gamma$ of positive measure; so does the normal derivative $\partial_\nu u$, by assumption.
Step 4. As a consequence, from identities (6) again, both $\partial_r U$ and $\partial_\theta U$ vanish on the same subset of $\mathbb{T}$ with positive measure. Because $F' \in H^2$ from Step 2, both Cauchy-Riemann equations and formula (4) hold true up to the boundary where

$$F'(e^{i\theta}) = -i \, e^{-i\theta} \, (\partial_\theta U + i \partial_r U) \, (e^{i\theta}) \, ,$$

from [8, Thm 3.11], and we conclude that F' vanish on some subset of $\mathbb{T}$ with positive measure.
Once applied to $G = F' \in H^2$, Theorem 3.1 implies that $F' \equiv 0$ in $\mathbb{D}$, then $\partial_{x_1} U = \partial_{x_2} U = 0$ in $\mathbb{D}$, that is U is a constant. Such constant is 0 since u vanishes on γ. Eventually, u vanishes in Ω.
Proposition 1 ensures that the tangential derivative of a harmonic function on the boundary is a well defined L^2 function as soon as the tangential derivative of its conjugate function, that is the normal derivative on the boundary, is assumed to be a L^2 function. This regularity result is established in Dini-smooth domains where conformal mappings are C^1-smooth with non vanishing derivative.

4 Extensions

It turns out to be more complicated when we assume the domain to be Lipschitz-smooth only, because in this case the conformal mapping may no longer be smooth. Then, the regularity result of Proposition 1 still holds, and relies on the so-called Muckenhoupt condition A_2, which is satisfied by the boundary values of the conformal mapping. The extension from Dini-smooth to Lipschitz domains is detailed in [2].
The extension of Theorems 2.2 and 2.3 from the Laplace equation to the conductivity equation with conductivity in $W^{1,r}$, for $r > 2$, is also addressed in [2]. In this case it is proved that the complex derivative of the solution to the conductivity

equation can be factorized, the trace on the boundary of one of the two factors being a non vanishing function, the other one being a Hardy function on which result of Theorem 3.1 can be applied. We provide in [2] a quite complete description of the corresponding Hardy-Smirnov classes of generalized holomorphic functions and of their links with the solutions to the conductivity equation.
Last, a natural question is whether Theorems 2.2 and 2.3 remain true or not in $\mathbb{R}^n$ for higher dimension $n \geq 3$. Counterexamples to unique continuation on the unit ball are derived in [2] from the famous one in [3]. They are to the effect that these theorems do not hold anymore when $n \geq 3$.

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