

ON SIMULTANEOUS IDENTIFICATION OF A SCATTERER AND ITS GENERALIZED IMPEDANCE BOUNDARY CONDITION

LAURENT BOURGEOIS*, NICOLAS CHAULET †, AND HUSSEM HADDAR ‡

Abstract. We consider the inverse scattering problem consisting in the identification of both an obstacle and its “equivalent impedance” from farfield measurements at a fixed frequency. The first specificity of this work is to consider the cases where this impedance is not a scalar function but a second order surface operator. The latter can be seen as a more general model for effective impedances and is for instance widely used for scattering from thin coatings. The second specificity of this work is to characterize the derivative of a least square cost functional with respect to this complex configuration. We provide in particular an extension of the notion of shape derivative to the cases where the impedance parameters cannot be considered as the traces of given functions. For instance, the obtained derivative does not vanish (in general) for tangential perturbations. The efficiency of considering this type of derivative is illustrated by some 2D numerical experiments based on a (classical) steepest descent method. The feasibility of retrieving both the obstacle and the impedance functionals is discussed in further numerical experiments.

Key words. Inverse scattering problem, Helmholtz equation, Generalized Impedance Boundary Conditions, Fréchet derivative, Steepest descent method.

AMS subject classifications.

1. Introduction. The imaging and identification of complex targets from measurements of scattered waves is a classical problem in inverse scattering theory that has considerable interest in many real life domains, especially those related to radar or sonar applications. The underlying inverse problem is challenging due to the inherent ill posedness and also the non linearity of the problem. This is why the use of simplified model in the design of numerical methods is a privileged route, whenever this can be made possible. One class of these simplified models is the so-called Generalized Impedance Boundary Conditions (GIBC). It is for instance widely used in the modelling of imperfectly absorbing targets or coated ones. The simplest version of these models corresponds with the so-called impedance or (Robin-Fourier) boundary condition. However, as demonstrated in many recent studies (see [4, 11, 12, 13, 2]), this simplified version cannot satisfactorily model the cases of either multi-angle illuminations or coated (and/or corrugated) surfaces. A more acceptable simplified model would rather correspond with an impedance in the form of a second order surface operator. The goal of the present work is to address the above mentioned inverse problem using this type of models. More precisely we shall consider the scalar problem (modelling either acoustic waves, or electromagnetic waves in a 2D setting) and consider GIBC of the form

$$\frac{\partial u}{\partial \nu} + \operatorname{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \Gamma,$$

where Γ is the boundary of an obstacle D , μ and λ are complex valued functions, $\operatorname{div}_\Gamma$ and ∇_Γ are respectively the surface divergence and the surface gradient on Γ and ν denotes the unit normal to Γ directed to the exterior of D . As a simple example, if the scatterer is a perfect conductor coated with a thin dielectric layer, for the transverse electric polarization, the GIBC model corresponds with $\mu = \delta$ and $\lambda = \delta k^2 n$, where k denotes the wavenumber, δ is the (possibly non constant) width of the layer and n is its refractive index [2].

The inverse problem under investigation is then the identification of both the obstacle D and the coefficients λ and μ from farfields measurements at a fixed frequency. The case $\mu = 0$ (that corresponds with classical impedance boundary conditions) has been addressed for instance in [24, 21, 15] and in [23, 3, 25, 9] for the Laplace equation. The case $\mu \neq 0$ has been considered in [8, 7] where the main interest is to characterize the impedance operator from only few measurements

*Laboratoire POEMS, ENSTA, 32, Boulevard Victor, 75739 Paris Cedex 15, France
(laurent.bourgeois@ensta.fr)

†CMAP, INRIA Saclay Ile de France et Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France

‡CMAP, INRIA Saclay Ile de France et Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France

(assuming a priori, perfect or imperfect knowledge of the surface) and study related stability issues. In the present work, the geometry is unknown and we authorize the use of many measurements corresponding to differently many choices of the incident plane waves.

We first prove the uniqueness of reconstructing both D and (λ, μ) when one uses incident plane waves with all possible directions and measures the farfields at all possible observation directions. The proof uses the technique of mixed reciprocity relation as in [18, 20].

Our second and main contribution is the characterization of the derivative of the farfield with respect to the domain D when the impedance parameters (λ, μ) are unknown. At a first glance, this may appear as a simple exercise using shape derivative tools, as described for example in the monographs [1, 16, 26]. However, it turned out that two specificities of the problem seriously complicate this task. The first one is indeed related to the second-order surface operator appearing in the expression of the impedance operator. As one could later notice, it leads to complex calculations and non intuitive final expressions of the derivatives.

The second major issue is not specific to GIBC but rather to the fact that the coefficients λ and/or μ are unknown functions supported by the unknown boundary ∂D . This configuration makes for instance the notion of partial derivative with respect to D unclear. To overcome this ambiguity we shall adapt the usual definition of partial derivative with respect to D by defining an appropriate extension of the impedance parameters λ and μ to a boundary $\partial D_\varepsilon = \partial D + \varepsilon(\partial D)$, where ε is a perturbation of ∂D (see definition 4.1 hereafter). As a surprising result, the obtained derivative depends not only on the normal part of the perturbation ε but also on the tangential part (see theorem 4.8 hereafter). In order to derive these results we adopted a method based on integral representation of the solutions as introduced in [19] for the case of Dirichlet or Neumann boundary conditions and in [14] for the case of constant impedance boundary conditions. An alternative method would have been the use of shape derivative tools, but probably with more involving technicalities.

In a last part of this work we shall present a numerical algorithm based on a classical least square formulation of the problem and an optimization technique based on a steepest descent method with regularization of the descent direction. The forward problem is solved by using a finite element method and the reconstruction update is based on a boundary variation technique (requiring a remesh of the computational domain at each step). The goal of this numerical section is to first illustrate the efficiency of considering the proposed non conventional type of shape derivative. Second, we present numerical experiments that discuss the feasibility of retrieving both the obstacle and the impedance functionals.

The outline of the paper is as follows. We describe the inverse problem in Section 2. In Section 3 we prove our uniqueness result, while Section 4 is dedicated to the evaluation of the derivative of the farfield with respect to the obstacle. In Section 5 we describe the optimization technique based on a least square formulation of the inverse problem. Some numerical tests in $2D$ showing the efficiency of the proposed steepest descent method are presented in Section 6. A technical lemma related to some differential geometry identities is presented in an Appendix.

2. The statement of the inverse problem. Let D be an open bounded domain of \mathbb{R}^d , with $d = 2$ or 3 , the boundary ∂D of which is Lipschitz continuous, such that $\Omega = \mathbb{R}^d \setminus \overline{D}$ is connected and let $(\lambda, \mu) \in (L^\infty(\partial D))^2$ be some impedance coefficients. The scattering problem with generalized impedance boundary conditions (GIBC) consists in finding $u = u^s + u^i$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \operatorname{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 & \text{on } \partial D \\ \lim_{R \rightarrow \infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - i k u^s \right|^2 ds = 0. \end{cases} \quad (2.1)$$

Here k is the wave number, $u^i = e^{ik\hat{d}\cdot x}$ is an incident plane wave where \hat{d} belongs to the unit sphere of \mathbb{R}^d denoted S^{d-1} , and $u^s \in V(\Omega) := \{v \in \mathcal{D}'(\Omega), \varphi v \in H^1(\Omega) \forall \varphi \in \mathcal{D}(\mathbb{R}^d) \text{ and } v|_{\partial D} \in H^1(\partial D)\}$ is the scattered field.

The surface operators $\operatorname{div}_\Gamma$ and ∇_Γ are precisely defined in Chapter 5 of [16]. For $v \in H^1(\partial D)$ the surface gradient $\nabla_\Gamma v$ lies in $L^2_\Gamma(\partial D) := \{V \in L^2(\partial D, \mathbb{R}^d), V \cdot \nu = 0\}$ while $\operatorname{div}_\Gamma(\mu \nabla_\Gamma u)$ is defined in $H^{-1}(\partial D)$ for $\mu \in L^\infty(\partial D)$ by

$$\langle \operatorname{div}_\Gamma(\mu \nabla_\Gamma u), v \rangle_{H^{-1}(\partial D), H^1(\partial D)} := - \int_{\partial D} \mu \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds \quad \forall v \in H^1(\partial D). \quad (2.2)$$

The last equation in (2.1) is the classical Sommerfeld radiation condition. The proof for well-posedness of problem (2.1) and the numerical computation of its solution can be done using the so-called Dirichlet-to-Neumann map so that we can give an equivalent formulation of (2.1) in a bounded domain $\Omega_R = \Omega \cap B_R$ where B_R is the ball of radius R such that $D \subset B_R$. The Dirichlet-to-Neumann map, $S_R : H^{1/2}(\partial B_R) \mapsto H^{-1/2}(\partial B_R)$ is defined for $g \in H^{1/2}(\partial B_R)$ by $S_R g := \partial u^e / \partial r|_{\partial B_R}$ where $u^e \in V(\mathbb{R}^d \setminus \overline{B_R})$ is the radiating solution of the Helmholtz equation outside B_R and $u^e = g$ on ∂B_R .

Solving (2.1) is equivalent to find u in $V_R := \{v \in H^1(\Omega_R); v|_{\partial D} \in H^1(\partial D)\}$ such that:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_R \\ \frac{\partial u}{\partial \nu} + \operatorname{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 & \text{on } \partial D \\ \frac{\partial u}{\partial r} - S_R(u) = \frac{\partial u^i}{\partial r} - S_R(u^i) & \text{on } \partial B_R. \end{cases} \quad (2.3)$$

We introduce the assumption

ASSUMPTION 1. *The coefficients $(\lambda, \mu) \in (L^\infty(\partial D))^2$ are such that*

$$\Im m(\lambda) \geq 0, \quad \Im m(\mu) \leq 0 \quad \text{a.e. in } \partial D$$

and there exists $c > 0$ such that

$$\Re e(\mu) \geq c \quad \text{a.e. in } \partial D.$$

Well-posedness of problem (2.3) is established in the following theorem, the proof of which is classical and given in [5].

THEOREM 2.1. *With assumption 1 the problem (2.3) has a unique solution u in V_R .*

In order to define the inverse problem, we recall now the definition of the farfield associated to a scattered field. From [10], the scattered field has the asymptotic behaviour:

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right) \right) \quad r \longrightarrow +\infty$$

uniformly for all the directions $\hat{x} = x/r \in S^{d-1}$ with $r = |x|$, and the farfield $u^\infty \in L^2(S^{d-1})$ has the following integral representation on the boundary ∂D :

$$u^\infty(\hat{x}) = \int_{\partial D} \left(u^s(y) \frac{\partial \Phi^\infty(\hat{x}, y)}{\partial \nu(y)} - \frac{\partial u^s(y)}{\partial \nu} \Phi^\infty(\hat{x}, y) \right) ds(y) \quad \forall \hat{x} \in S^{d-1}. \quad (2.4)$$

Here $\Phi^\infty(\cdot, y)$ is the farfield associated with the Green function $\Phi(\cdot, y)$ of the Helmholtz equation. The function $\Phi(\cdot, y)$ is defined in \mathbb{R}^2 by $\Phi(x, y) = (i/4)H_0^1(k|x-y|)$, where H_0^1 is the Hankel function of the first kind and of order 0, and in \mathbb{R}^3 by $e^{ik|x-y|}/(4\pi|x-y|)$. The associated farfields are defined in S^1 by $(e^{i\pi/4}/\sqrt{8\pi k})e^{-iky \cdot \hat{x}}$ and in S^2 by $(1/4\pi)e^{-iky \cdot \hat{x}}$ respectively. The second integral in (2.4) has to be understood as a duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$. We are now in a position to define the farfield map

$$T : (\lambda, \mu, \partial D) \rightarrow u^\infty$$

where u^∞ is the farfield associated with the scattered field $u^s = u - u^i$ and u is the unique solution of problem (2.1) with obstacle D and impedances (λ, μ) on ∂D .

The general inverse problem we are interested in is the following: given several incident plane waves of direction $\hat{d} \in S^{d-1}$, is it possible to reconstruct the obstacle D as well as the impedances λ and μ defined on ∂D from the corresponding farfield $u^\infty = T(\lambda, \mu, \partial D)$? The first question of interest is the identifiability of $(\lambda, \mu, \partial D)$ from the farfield data u^∞ , that is uniqueness.

3. A uniqueness result. In this section, we provide a uniqueness result concerning identification of both the obstacle D and the impedances (λ, μ) from the farfields associated to plane waves with all incident directions $\hat{d} \in S^{d-1}$. In this respect we denote by $u^\infty(\hat{x}, \hat{d})$ the farfield in the \hat{x} direction that is associated to the plane wave with direction \hat{d} . In the following, we introduce some regularity assumptions for the obstacle D and the impedances λ, μ .

ASSUMPTION 2. *The boundary ∂D is C^2 , and the impedances satisfy $\lambda \in C^0(\partial D)$ and $\mu \in C^1(\partial D)$.*

The main result is the following theorem, which is a generalization of the uniqueness result for $\mu = 0$ proved in [20].

THEOREM 3.1. *Assume that $(\lambda_1, \mu_1, \partial D_1)$ and $(\lambda_2, \mu_2, \partial D_2)$ satisfy assumptions 1 and 2, and the corresponding farfields $u_1^\infty = T(\lambda_1, \mu_1, \partial D_1)$ and $u_2^\infty = T(\lambda_2, \mu_2, \partial D_2)$ satisfy $u_1^\infty(\hat{x}, \hat{d}) = u_2^\infty(\hat{x}, \hat{d})$ for all $\hat{x} \in S^{d-1}$ and $\hat{d} \in S^{d-1}$. Then $D_1 = D_2$ and $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.*

The proof of the above theorem is based on several results, the first one is the mixed reciprocity lemma and does not require the regularity assumption 2.

LEMMA 3.2. *Let $w^\infty(\cdot, z)$ be the farfield associated to the incident field $\Phi(\cdot, z)$ with $z \in \Omega$, and $u^s(\cdot, \hat{x})$ be the scattered field associated to the plane wave of direction $\hat{x} \in S^{d-1}$. Then*

$$w^\infty(-\hat{x}, z) = c(d) u^s(z, \hat{x}),$$

with $c(2) = e^{i\pi/4}/\sqrt{8\pi k}$ and $c(3) = 1/4\pi$.

Proof. For two incident fields u_1^i and u_2^i , the associated total fields u_1 and u_2 satisfy, by using the boundary condition on ∂D ,

$$\begin{aligned} & \int_{\partial D} \left(u_1 \frac{\partial u_2}{\partial \nu} - u_2 \frac{\partial u_1}{\partial \nu} \right) ds \\ &= \int_{\partial D} (\mu \nabla_\Gamma u_1 \cdot \nabla_\Gamma u_2 - \lambda u_1 u_2 - \mu \nabla_\Gamma u_1 \cdot \nabla_\Gamma u_2 + \lambda u_1 u_2) ds = 0. \end{aligned}$$

By using the decomposition $u_1 = u_1^i + u_1^s$ and $u_2 = u_2^i + u_2^s$, that the incident fields solve the Helmholtz equation inside D , that the scattered fields solve the Helmholtz equation outside D as well as the radiation condition, we obtain

$$\int_{\partial D} \left(u_1^s \frac{\partial u_2^i}{\partial \nu} - u_2^i \frac{\partial u_1^s}{\partial \nu} \right) ds = \int_{\partial D} \left(u_2^s \frac{\partial u_1^i}{\partial \nu} - u_1^i \frac{\partial u_2^s}{\partial \nu} \right) ds. \quad (3.1)$$

Now we use the Green's formula on the boundary ∂D for $u^s(\cdot, \hat{x})$: for $z \in \Omega$ and $\hat{x} \in S^{d-1}$,

$$u^s(z, \hat{x}) = \int_{\partial D} \left(u^s(y, \hat{x}) \frac{\partial \Phi(y, z)}{\partial \nu(y)} - \frac{\partial u^s(y, \hat{x})}{\partial \nu} \Phi(y, z) \right) ds(y).$$

By applying equation (3.1) when u_1^i is the plane wave of direction \hat{x} and u_2^i is the point source $\Phi(\cdot, z)$, it follows that

$$u^s(z, \hat{x}) = \int_{\partial D} \left(w^s(y, z) \frac{\partial e^{ik\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial w^s(y, z)}{\partial \nu(y)} e^{ik\hat{x}\cdot y} \right) ds(y).$$

Lastly, from the integral representation (2.4) and the above equation we obtain

$$c(d) u^s(z, \hat{x}) = w^\infty(-\hat{x}, z).$$

□

The second lemma is a density result and does not require the regularity assumption 2 either. Since it is a slightly more general version of lemma 4 in [8], the proof is omitted.

LEMMA 3.3. *Let $u(\cdot, \hat{d})$ denote the solution of (2.1) associated to the incident wave $u^i(x) = e^{ik\hat{d}\cdot x}$ and assume that for some $f \in H^{-1}(\partial D)$,*

$$\langle u(\cdot, \hat{d}), f \rangle_{H^1(\partial D), H^{-1}(\partial D)} = 0, \quad \forall \hat{d} \in S^{d-1}.$$

Then $f = 0$.

We are now in a position to prove theorem 3.1.

Proof. [Proof of theorem 3.1] The first step of the proof consists in proving that $D_1 = D_2$, following the method of [18, 17]. Let us denote $\tilde{\Omega}$ the unbounded connected component of $\mathbb{R}^d \setminus \overline{D_1 \cup D_2}$. From Rellich's lemma and unique continuation, we obtain that

$$u_1^s(z, \hat{d}) = u_2^s(z, \hat{d}), \quad \forall z \in \tilde{\Omega}, \forall \hat{d} \in S^{d-1}. \quad (3.2)$$

Using the mixed reciprocity lemma 3.2, we obtain that

$$u_1^\infty(-\hat{d}, z) = u_2^\infty(-\hat{d}, z), \quad \forall \hat{d} \in S^{d-1}, \forall z \in \tilde{\Omega},$$

where $u_1^\infty(\cdot, z)$ and $u_2^\infty(\cdot, z)$ are the farfields associated to the incident field $\Phi(\cdot, z)$ with $z \in \tilde{\Omega}$. By using again Rellich's lemma and unique continuation, it follows that

$$u_1^s(x, z) = u_2^s(x, z), \quad \forall (x, z) \in \tilde{\Omega} \times \tilde{\Omega}. \quad (3.3)$$

Assume that $D_1 \not\subset D_2$. Since $\mathbb{R}^d \setminus \overline{D_2}$ is connected, there exists some non empty open set $\Gamma_* \subset (\partial D_1 \cap \partial \tilde{\Omega}) \setminus \overline{D_2}$. We now consider some point $x_* \in \Gamma_*$ and the sequence

$$x_n = x_* + \frac{\nu_1(x_*)}{n}.$$

For sufficiently large n , $x_n \in \tilde{\Omega}$. From (3.3), we hence have by denoting $P_1 v := \partial v / \partial \nu + \text{div}_\Gamma(\mu_1 \nabla_\Gamma v) + \lambda_1 v$,

$$P_1 u_2^s(\cdot, x_n) = P_1 u_1^s(\cdot, x_n) \quad \text{on } \Gamma_*.$$

Using boundary condition on ∂D_1 for $u_1 = u_1^s + \Phi(\cdot, x_n)$, this implies that

$$P_1 u_2^s(\cdot, x_n) = -P_1 \Phi(\cdot, x_n) \quad \text{on } \Gamma_*.$$

Using assumption 2 and the fact that u_2^s is smooth in the neighborhood of Γ_* , we obtain

$$\lim_{n \rightarrow +\infty} P_1 u_2^s(\cdot, x_n) = \frac{\partial u_2^s}{\partial \nu}(\cdot, x_*) + \mu_1 \Delta_\Gamma u_2^s(\cdot, x_*) + \nabla_\Gamma \mu_1 \cdot \nabla_\Gamma u_2^s(\cdot, x_*) + \lambda_1 u_2^s(\cdot, x_*)$$

in $L^2(\Gamma_*)$. On the other hand, for $x_1 \in \Gamma_* \setminus \{x_*\}$, we have pointwise convergence

$$\lim_{n \rightarrow +\infty} P_1 \Phi(x_1, x_n) = \frac{\partial \Phi}{\partial \nu}(x_1, x_*) + \mu_1 \Delta_\Gamma \Phi(x_1, x_*) + \nabla_\Gamma \mu_1 \cdot \nabla_\Gamma \Phi(x_1, x_*) + \lambda_1 \Phi(x_1, x_*).$$

We hence obtain that

$$\frac{\partial \Phi}{\partial \nu}(\cdot, x_*) + \text{div}_\Gamma(\mu_1 \nabla_\Gamma \Phi)(\cdot, x_*) + \lambda_1 \Phi(\cdot, x_*) \in L^2(\Gamma_*). \quad (3.4)$$

Now we consider some reals $R_* > r_* > 0$ such that $\partial D \cap B(x_*, R_*) \subset \Gamma_*$, a function $\phi \in C_0^\infty(B(x_*, R_*))$ with $\phi = 1$ on $\overline{B(x_*, r_*)}$, and $w_*^s := \phi \Phi(\cdot, x_*)$. The function w_*^s satisfies

$$\begin{cases} \Delta w_*^s + k^2 w_*^s = f & \text{in } \Omega_1 \\ \frac{\partial w_*^s}{\partial \nu} + \text{div}_\Gamma(\mu_1 \nabla_\Gamma w_*^s) + \lambda_1 w_*^s = g & \text{on } \partial D_1 \\ \lim_{R \rightarrow \infty} \int_{|x|=R} \left| \frac{\partial w_*^s}{\partial r} - i k w_*^s \right|^2 ds = 0, \end{cases} \quad (3.5)$$

with $\Omega_1 = \mathbb{R}^d \setminus \overline{D_1}$ and

$$\begin{aligned} f &= (\Delta \phi) \Phi(\cdot, x_*) + 2 \nabla \phi \cdot \nabla \Phi(\cdot, x_*), \\ g &= \phi \left(\frac{\partial \Phi}{\partial \nu}(\cdot, x_*) + \text{div}_\Gamma(\mu_1 \nabla_\Gamma \Phi)(\cdot, x_*) + \lambda_1 \Phi(\cdot, x_*) \right) \\ &\quad + \Phi(\cdot, x_*) \left(\frac{\partial \phi}{\partial \nu} + \nabla_\Gamma \mu_1 \cdot \nabla_\Gamma \phi + \mu_1 \Delta_\Gamma \phi \right) + 2 \mu_1 \nabla_\Gamma \Phi(\cdot, x_*) \cdot \nabla_\Gamma \phi. \end{aligned}$$

Since $\phi = 1$ in the neighborhood of x_* and by using (3.4), we have $f \in L^2(\Omega_1)$ and $g \in L^2(\partial D_1)$. With the help of a variational formulation for the auxiliary problem (3.5) as in [7], we conclude that $w_*^s \in H^1(B_R \setminus \overline{D_1})$, hence $\Phi(\cdot, x_*) \in H^1(\Omega_1 \cap B(x_*, r_*))$. Since ∂D is C^2 , we can find a finite cone C_* of apex x_* , angle θ_* , radius r_* and axis directed by $\xi_* = \nu_1(x_*)$, such that $C_* \subset \Omega_1 \cap B(x_*, r_*)$. Hence $\Phi(\cdot, x_*) \in H^1(C_*)$.

In the case $d = 3$ (the case $d = 2$ is similar), we have

$$\nabla \Phi(\cdot, x_*) = -\frac{e^{ik|x-x_*|}}{4\pi|x-x_*|^2} \left(\frac{1}{|x-x_*|} - ik \right) (x-x_*),$$

and by using spherical coordinates (r, θ, ϕ) centered at x_* ,

$$\int_{C_*} \frac{dx}{|x-x_*|^4} = \int_0^{r_*} \int_0^{\theta_*} \int_0^{2\pi} \frac{r^2 \sin \theta dr d\theta d\phi}{r^4} = +\infty,$$

which contradicts the fact that $\Phi(\cdot, x_*) \in H^1(C_*)$. Then $D_1 \subset D_2$. We prove the same way that $D_2 \subset D_1$, and then $D_1 = D_2 = D$.

The second step of the proof consists in proving that $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$. In this view we denote $\lambda = \lambda_1 - \lambda_2$ and $\mu = \mu_1 - \mu_2$. From equality (3.2), the total fields associated with the plane waves of direction \hat{d} satisfy

$$u(x, \hat{d}) := u_1(x, \hat{d}) = u_2(x, \hat{d}) \quad \forall x \in \mathbb{R}^d \setminus \overline{D}, \forall \hat{d} \in S^{d-1}.$$

From the boundary condition on ∂D for u_1 and u_2 it follows that

$$\operatorname{div}_\Gamma(\mu \nabla_\Gamma u(\cdot, \hat{d})) + \lambda u(\cdot, \hat{d}) = 0 \quad \text{on } \partial D, \quad \forall \hat{d} \in S^{d-1}.$$

For some $\phi \in H^1(\partial D)$, multiplying the above equation with integration by parts leads to

$$\langle u(\cdot, \hat{d}), \operatorname{div}_\Gamma(\mu \nabla_\Gamma \phi) + \lambda \phi \rangle_{H^1(\partial D), H^{-1}(\partial D)} = 0, \quad \forall \hat{d} \in S^{d-1}.$$

With the help of lemma 3.3, we obtain that

$$\operatorname{div}_\Gamma(\mu \nabla_\Gamma \phi) + \lambda \phi = 0 \quad \text{on } \partial D, \quad \forall \phi \in H^1(\partial D).$$

Choosing $\phi = 1$ in the above equation leads to $\lambda = 0$. The above equation also implies that

$$\int_{\partial D} \mu |\nabla_\Gamma \phi|^2 ds = 0, \quad \forall \phi \in H^1(\partial D).$$

Assume that $\mu(x_0) \neq 0$ for some $x_0 \in \partial D$, then for example $\Re e(\mu)(x_0) > 0$ without loss of generality. Since μ is continuous there exists $\varepsilon > 0$ such that $\Re e(\mu)(x) > 0$ for all $x \in \partial D \cap B(x_0, \varepsilon)$. Let us choose ϕ as a smooth and compactly supported function in $\partial D \cap B(x_0, \varepsilon)$. We obtain that

$$\int_{\partial D \cap B(x_0, \varepsilon)} \Re e(\mu) |\nabla_\Gamma \phi|^2 ds = 0,$$

and then $\nabla_\Gamma \phi = 0$ on ∂D , that is ϕ is a constant on ∂D , which is a contradiction. We hence have $\mu = 0$ on ∂D , which completes the proof. \square

As illustrated by theorem 3.1, if all plane waves are used as incident fields, then it is possible to retrieve both the obstacle and the impedances, with reasonable assumptions on such unknowns. In the sequel, we consider an effective method to retrieve both the obstacle and the impedances in the case we use several plane waves. Such method will be based on a standard steepest descent method and in this view, we need to compute the partial derivative of the farfield with respect to the obstacle, the impedances being fixed. This is the aim of next section. The computation of the partial derivative with respect to the impedances is already known and given in [7]. The adopted approach is the one used in [19] for the Neumann boundary condition and in [14] for the classical impedance boundary condition with constant λ .

4. Differentiation of farfield with respect to the obstacle. Throughout this section, we assume that the boundary of the obstacle and the impedances are smooth, typically ∂D is C^4 , $\lambda \in C^2(\partial D)$ and $\mu \in C^3(\partial D)$, which ensures that the solution to problem (2.3) belongs to $H^4(\Omega_R)$. In order to compute the partial derivative of the farfield associated to the solution of problem (2.1) with respect to the obstacle, we consider a perturbed obstacle D_ε and some impedances $(\lambda_\varepsilon, \mu_\varepsilon)$ that correspond to the impedances (λ, μ) once transported on the perturbed boundary ∂D_ε . More precisely, we consider some mapping $\varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ with $C^{1,\infty} := C^1 \cap W^{1,\infty}$ equipped with the norm $\|\varepsilon\| := \|\varepsilon\|_{W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)}$. From [16, section 5.2.2], if we assume that $\|\varepsilon\| < 1$, the mapping $f_\varepsilon := Id + \varepsilon$ is a C^1 -diffeomorphism of \mathbb{R}^d . The perturbed obstacle D_ε is defined from D by

$$\partial D_\varepsilon = \{x + \varepsilon(x), x \in \partial D\},$$

while the transported impedances $(\lambda_\varepsilon, \mu_\varepsilon)$ on ∂D_ε are defined from (λ, μ) by

$$\lambda_\varepsilon = \lambda \circ f_\varepsilon^{-1}, \quad \mu_\varepsilon = \mu \circ f_\varepsilon^{-1}. \quad (4.1)$$

We now define the partial derivative of the farfield with respect to the obstacle.

DEFINITION 4.1. *We say that the farfield operator $T : (\lambda, \mu, \partial D) \rightarrow u^\infty$ is differentiable with respect to ∂D if there exists a continuous linear operator $T'_{\lambda,\mu}(\partial D) : C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \rightarrow L^2(S^{d-1})$ and a function $o(\|\varepsilon\|) : C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \rightarrow L^2(S^{d-1})$ such that*

$$T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D) = T'_{\lambda,\mu}(\partial D) \cdot \varepsilon + o(\|\varepsilon\|),$$

where λ_ε and μ_ε are defined by (4.1) and $\lim_{\|\varepsilon\| \rightarrow 0} o(\|\varepsilon\|)/\|\varepsilon\| = 0$ in $L^2(S^{d-1})$.

REMARK 1. *Note that if λ and μ are constants, the above definition coincides with the classical notion of Fréchet differentiability with respect to an obstacle.*

Now we denote by u_ε the solution of problem (2.1) with obstacle D_ε instead of obstacle D and impedances $(\lambda_\varepsilon, \mu_\varepsilon)$ instead of impedances (λ, μ) . We assume in addition that $\overline{D} \subset D_\varepsilon$. We have the following integral representation for $u_\varepsilon^s - u^s$:

LEMMA 4.2. *For $x \in \mathbb{R}^d \setminus \overline{D}_\varepsilon$,*

$$u_\varepsilon^s(x) - u^s(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left\{ \frac{\partial w}{\partial \nu_\varepsilon}(\cdot, x) + \operatorname{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot, x) + \lambda_\varepsilon w(\cdot, x) \right\} ds_\varepsilon,$$

where $w(\cdot, x)$ is the solution of problem (2.1) with incident wave $\Phi(\cdot, x)$.

Proof. Let $x \in \mathbb{R}^d \setminus \overline{D}_\varepsilon$. By using the Green's integral theorem inside D for plane wave u^i and point source $\Phi(\cdot, x)$, we have

$$\int_{\partial D} \left(u^i \frac{\partial \Phi}{\partial \nu}(\cdot, x) - \frac{\partial u^i}{\partial \nu} \Phi(\cdot, x) \right) ds = 0,$$

then obtain the representation formula

$$u^s(x) = \int_{\partial D} \left(u \frac{\partial \Phi}{\partial \nu}(\cdot, x) - \frac{\partial u}{\partial \nu} \Phi(\cdot, x) \right) ds. \quad (4.2)$$

By using the boundary condition for functions u , $w(\cdot, x)$ and formula (2.2), we obtain

$$\int_{\partial D} \left(u \frac{\partial \Phi}{\partial \nu}(\cdot, x) - \frac{\partial u}{\partial \nu} \Phi(\cdot, x) \right) ds = - \int_{\partial D} \left(u \frac{\partial w^s}{\partial \nu}(\cdot, x) - \frac{\partial u}{\partial \nu} w^s(\cdot, x) \right) ds.$$

By using again the Green's integral theorem outside D and the radiation condition for u^s and w^s , we obtain

$$u^s(x) = \int_{\partial D} \left(\frac{\partial u^i}{\partial \nu} w^s(\cdot, x) - u^i \frac{\partial w^s}{\partial \nu}(\cdot, x) \right) ds.$$

We now use the Green's integral theorem in $D_\varepsilon \setminus \overline{D}$ and find

$$w^s(x) = \int_{\partial D_\varepsilon} \left(\frac{\partial u^i}{\partial \nu_\varepsilon} w^s(\cdot, x) - u^i \frac{\partial w^s}{\partial \nu_\varepsilon}(\cdot, x) \right) ds_\varepsilon.$$

Using again the Green's integral theorem outside D_ε and the radiation condition for u_ε^s and w^s , we obtain

$$u^s(x) = \int_{\partial D_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} w^s(\cdot, x) - u_\varepsilon \frac{\partial w^s}{\partial \nu_\varepsilon}(\cdot, x) \right) ds_\varepsilon.$$

The boundary condition satisfied by u_ε on ∂D_ε implies

$$-u^s(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left(\frac{\partial w^s}{\partial \nu_\varepsilon}(\cdot, x) + \operatorname{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w^s)(\cdot, x) + \lambda_\varepsilon w^s(\cdot, x) \right) ds_\varepsilon.$$

Lastly, we use formula (4.2) for u_ε and D_ε , as well as the boundary condition of u_ε on ∂D_ε , and obtain that for $x \in \mathbb{R}^d \setminus \overline{D}$,

$$u_\varepsilon^s(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left(\frac{\partial \Phi}{\partial \nu_\varepsilon}(\cdot, x) + \operatorname{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} \Phi)(\cdot, x) + \lambda_\varepsilon \Phi(\cdot, x) \right) ds.$$

We complete the proof by adding the two last equations, given $w(\cdot, x) = w^s(\cdot, x) + \Phi(\cdot, x)$. \square

We continue our computation by replacing u_ε by u in the integral representation of lemma 4.2 at first order for $\|\varepsilon\|$, uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

LEMMA 4.3. *We have*

$$u_\varepsilon^s(x) - u^s(x) = \int_{\partial D_\varepsilon} u \left\{ \frac{\partial w}{\partial \nu_\varepsilon}(\cdot, x) + \operatorname{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot, x) + \lambda_\varepsilon w(\cdot, x) \right\} ds_\varepsilon + \mathcal{O}(\|\varepsilon\|^2),$$

uniformly for x in each compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

Proof. Here we only give a short proof, the complete one is detailed in [6]. In the following, we make use of the following definitions

$$J_\varepsilon := |\det(\nabla f_\varepsilon)|, \quad J'_\varepsilon := J_\varepsilon |(\nabla f_\varepsilon)^{-T} \nu|, \quad P_\varepsilon := (\nabla f_\varepsilon)^{-1} (\nabla f_\varepsilon)^{-T},$$

where $\det(B)$ stands for the determinant of matrix B , while B^{-T} stands for the transposition of the inverse of invertible matrix B .

We have

$$\begin{aligned} & \int_{\partial D_\varepsilon} (u_\varepsilon - u) \left\{ \frac{\partial w}{\partial \nu_\varepsilon}(\cdot, x) + \operatorname{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot, x) + \lambda_\varepsilon w(\cdot, x) \right\} ds_\varepsilon \\ &= \int_{\partial D_\varepsilon} (u_\varepsilon - u) \frac{\partial w}{\partial \nu_\varepsilon} ds_\varepsilon - \int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon} (u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w ds_\varepsilon + \int_{\partial D_\varepsilon} \lambda_\varepsilon (u_\varepsilon - u) w ds_\varepsilon. \end{aligned} \quad (4.3)$$

We only consider the second integral, the others (which are simpler) are treated in [6]. By denoting $\tilde{u}_\varepsilon = u_\varepsilon \circ f_\varepsilon$, $\hat{u}_\varepsilon = u \circ f_\varepsilon$ and $\hat{w}_\varepsilon = w \circ f_\varepsilon$, the second integral becomes

$$\int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon} (u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w ds_\varepsilon = \int_{\partial D_\varepsilon} (\mu \circ f_\varepsilon^{-1}) \nabla_{\Gamma_\varepsilon} ((\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \circ f_\varepsilon^{-1}) \cdot \nabla_{\Gamma_\varepsilon} (\hat{w}_\varepsilon \circ f_\varepsilon^{-1}) ds_\varepsilon.$$

We may prove (see [7, proof of lemma 3.4]) that for $z \in H^1(\partial D)$, $x \in \partial D$ and $x_\varepsilon = f_\varepsilon(x)$,

$$\nabla_{\Gamma_\varepsilon} (z \circ f_\varepsilon^{-1})(x_\varepsilon) = (\nabla f_\varepsilon(x))^{-T} \nabla_\Gamma z(x).$$

As a consequence, the change of variable $x_\varepsilon = f_\varepsilon(x)$ in the integral (see [16, proposition 5.4.3]) implies

$$\int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon} (u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w ds_\varepsilon = \int_{\partial D} \mu \nabla_\Gamma (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \cdot P_\varepsilon \cdot \nabla_\Gamma \hat{w}_\varepsilon J'_\varepsilon ds,$$

and

$$\begin{aligned} & \int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon} (u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon - \int_{\partial D} \mu \nabla_{\Gamma} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \cdot \nabla_{\Gamma} w \, ds \\ &= \int_{\partial D} \mu \nabla_{\Gamma} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \cdot (J_\varepsilon^\nu P_\varepsilon \cdot \nabla_{\Gamma} \hat{w}_\varepsilon - \nabla_{\Gamma} w) \, ds. \end{aligned}$$

Then

$$\begin{aligned} & \left| \int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon} (u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon - \int_{\partial D} \mu \nabla_{\Gamma} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \cdot \nabla_{\Gamma} w \, ds \right| \\ & \leq \|\mu\|_{L^\infty(\partial D)} \|\tilde{u}_\varepsilon - \hat{u}_\varepsilon\|_{H^1(\partial D)} V_\varepsilon(x) \end{aligned}$$

with

$$V_\varepsilon(x) = \|(J_\varepsilon^\nu P_\varepsilon \cdot \nabla_{\Gamma} (w \circ f_\varepsilon) - \nabla_{\Gamma} w)(\cdot, x)\|_{L^2(\partial D)}.$$

By using the fact that

$$J_\varepsilon^\nu = 1 + \mathcal{O}(\|\varepsilon\|), \quad P_\varepsilon = Id(1 + \mathcal{O}(\|\varepsilon\|)),$$

we conclude that $V_\varepsilon(x) = \mathcal{O}(\|\varepsilon\|)$, uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

On the other hand,

$$\|\tilde{u}_\varepsilon - \hat{u}_\varepsilon\|_{H^1(\partial D)} \leq \|\tilde{u}_\varepsilon - u\|_{H^1(\partial D)} + \|\hat{u}_\varepsilon - u\|_{H^1(\partial D)}.$$

We have

$$\|\tilde{u}_\varepsilon - u\|_{H^1(\partial D)} = \mathcal{O}(\|\varepsilon\|), \quad \|\hat{u}_\varepsilon - u\|_{H^1(\partial D)} = \mathcal{O}(\|\varepsilon\|).$$

The first estimate is a consequence of theorem 3.1 in [7], that is continuity of the solution of problem (2.3) with respect to the scatterer D . The second one comes from the fact that $\hat{u}_\varepsilon = u \circ f_\varepsilon$.

We hence conclude that

$$\int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon} (u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w(\cdot, x) \, ds_\varepsilon - \int_{\partial D} \mu \nabla_{\Gamma} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \cdot \nabla_{\Gamma} w(\cdot, x) \, ds = \mathcal{O}(\|\varepsilon\|),$$

uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$, and we treat the other two integrals of (4.3) similarly. We remark that due to boundary condition satisfied by $w(\cdot, x)$ on ∂D , we have

$$\begin{aligned} 0 &= \int_{\partial D} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \frac{\partial w}{\partial \nu} \, ds - \int_{\partial D} \mu \nabla_{\Gamma} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \cdot \nabla_{\Gamma} w \, ds + \int_{\partial D} \lambda (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) w \, ds \\ &= \int_{\partial D_\varepsilon} (u_\varepsilon - u) \frac{\partial w}{\partial \nu_\varepsilon} \, ds_\varepsilon - \int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon} (u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon + \int_{\partial D_\varepsilon} \lambda_\varepsilon (u_\varepsilon - u) w \, ds_\varepsilon + \mathcal{O}(\|\varepsilon\|^2), \end{aligned}$$

and conclude that

$$\int_{\partial D_\varepsilon} (u_\varepsilon - u) \left\{ \frac{\partial w}{\partial \nu_\varepsilon}(\cdot, x) + \operatorname{div}_{\Gamma_\varepsilon} (\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot, x) + \lambda_\varepsilon w(\cdot, x) \right\} \, ds_\varepsilon = \mathcal{O}(\|\varepsilon\|^2),$$

which completes the proof in view of lemma 4.2. \square

To continue the computation of the partial derivative of solution u of problem (2.1) with respect to the domain D , we need to extend the definitions of some surface quantifies, essentially the outward normal ν on ∂D and the surface gradient ∇_{Γ} . inside the volumic domain $D_\varepsilon \setminus \overline{D}$. In this view, for $x_0 \in \partial D$, by definition of a domain of class C^1 there exist a function ϕ of class C^1 and two open sets $U \subset \mathbb{R}^{d-1}$ and $V \subset \mathbb{R}^d$ which are neighborhood of 0 and x_0 respectively, such that $\phi(0) = x_0$ and

$$\partial D \cap V = \{\phi(\xi); \xi \in U\}.$$

Defining now for $t \in [0, 1]$,

$$f_t := Id + t\varepsilon, \quad \phi_t := f_t \circ \phi,$$

ϕ_t is a parametrization of $\partial D_t = (Id + t\varepsilon)(\partial D)$, and hence the tangential vectors of ∂D_t at $x_0^t = f_t(x_0)$ are

$$e_j^t = \frac{\partial \phi_t}{\partial \xi_j} = (Id + t\nabla \varepsilon) \frac{\partial \phi}{\partial \xi_j} = (Id + t\nabla \varepsilon) e_j, \quad \text{for } j = 1, d-1. \quad (4.4)$$

To such basis we associate the covariant basis (e_i^t) of ∂D_t at point x_0^t (see for example [22, section 2.5]) by

$$e_i^t \cdot e_j^t = \delta_j^i, \quad \text{for } i, j = 1, d-1. \quad (4.5)$$

With these definitions, the outward normal of ∂D_t at point x_0^t is given by

$$\nu_t = \frac{e_1^t \times e_2^t}{|e_1^t \times e_2^t|},$$

while the tangential gradient of function $w \in H^1(\partial D_t)$ is given, denoting $\tilde{w}_t = w \circ \phi_t$, by

$$\nabla_{\Gamma_t} w(x_0^t) = \sum_{i=1}^{d-1} \frac{\partial \tilde{w}_t}{\partial \xi_i}(0) e_i^t. \quad (4.6)$$

It is hence possible to consider in domain $D_\varepsilon \setminus \overline{D}$ an extended outward normal ν_t and an extended tangential gradient $\nabla_{\Gamma_t} w$ by using parametrization (ξ_i, t) for $i = 1, d-1$. In the same spirit, the impedances (λ, μ) are extended to (λ_t, μ_t) , that is

$$\lambda_t = \lambda \circ f_t^{-1}, \quad \mu_t = \mu \circ f_t^{-1}.$$

We are now in a position to transform the integral representation of $u_\varepsilon^s - u^s$ on ∂D_ε in lemma 4.3 into an integral representation on ∂D . We have the following proposition.

PROPOSITION 4.4. *We have*

$$\begin{aligned} & u_\varepsilon^s(x) - u^s(x) \\ &= \int_{\partial D} (\varepsilon \cdot \nu) \operatorname{div} (-\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w(\cdot, x) \nu_t + u \nabla w(\cdot, x) + \lambda_t u w(\cdot, x) \nu_t)|_{t=0} ds + \mathcal{O}(\|\varepsilon\|^2), \end{aligned}$$

uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

Proof. The proof relies on the Green's integral theorem and on a change of variable. We have by using the extension of fields which is described above and remarking that $\nu^0 = \nu$ and $\nu^1 = \nu_\varepsilon$,

$$\begin{aligned} & \int_{\partial D_\varepsilon} \left\{ u \frac{\partial w}{\partial \nu_\varepsilon} - \mu_\varepsilon \nabla_{\Gamma_\varepsilon} u \cdot \nabla_{\Gamma_\varepsilon} w + \lambda_\varepsilon u w \right\} ds_\varepsilon - \int_{\partial D} \left\{ u \frac{\partial w}{\partial \nu} - \mu \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w + \lambda u w \right\} ds \\ &= \int_{D_\varepsilon \setminus \overline{D}} \operatorname{div} \{ u \nabla w - \mu_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \nu_t + \lambda_t u w \nu_t \} dx \\ &= \int_{\partial D} \int_0^1 \operatorname{div} \{ u \nabla w - \mu_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \nu_t + \lambda_t u w \nu_t \} (\varepsilon \cdot \nu) dt ds + \mathcal{O}(\|\varepsilon\|^2). \end{aligned}$$

Here we have used the change of variable $(x_{\partial D}, t) \rightarrow x_{\partial D} + t\varepsilon(x_{\partial D})$ for $x_{\partial D} \in \partial D$ and $t \in [0, 1]$, the determinant of the associated Jacobian matrix at first order being

$$J = (\varepsilon \cdot \nu) + t \mathcal{O}(\|\varepsilon\|^2),$$

as well as the fact that $(\varepsilon \cdot \nu) \geq 0$. Lastly, by using a first order approximation of the integrand as in [19], we obtain

$$\begin{aligned} & \int_{\partial D_\varepsilon} \left\{ u \frac{\partial w}{\partial \nu_\varepsilon} - \mu_\varepsilon \nabla_{\Gamma_\varepsilon} u \cdot \nabla_{\Gamma_\varepsilon} w + \lambda_\varepsilon u w \right\} ds_\varepsilon - \int_{\partial D} \left\{ u \frac{\partial w}{\partial \nu} - \mu \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w + \lambda u w \right\} ds \\ &= \int_{\partial D} (\varepsilon \cdot \nu) \operatorname{div} \{ u \nabla w - \mu_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \nu_t + \lambda_t u w \nu_t \} |_{t=0} ds + \mathcal{O}(\|\varepsilon\|^2). \end{aligned}$$

We complete the proof by using the boundary condition satisfied by u on ∂D , formula (2.2) and the result of lemma 4.3. \square

The remainder of the section consists in computing the divergence term. In order to do that we need the following technical lemma, the proof of which is postponed in an appendix.

LEMMA 4.5. *We have*

$$\begin{aligned} (\varepsilon \cdot \nu) (\nabla \lambda_t \cdot \nu_t) |_{t=0} &= -(\nabla_{\Gamma} \lambda \cdot \varepsilon), \\ (\operatorname{div} \nu_t) |_{t=0} &= \operatorname{div}_{\Gamma} \nu, \end{aligned}$$

and by denoting $\varepsilon_{\Gamma} = \varepsilon - (\varepsilon \cdot \nu) \nu$,

$$\begin{aligned} & (\varepsilon \cdot \nu) \nabla (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \cdot \nu_t |_{t=0} = -\varepsilon_{\Gamma} \cdot \nabla_{\Gamma} (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) \\ & + \nabla_{\Gamma} (\varepsilon_{\Gamma} \cdot \nabla_{\Gamma} u + (\nabla u \cdot \nu) (\varepsilon \cdot \nu)) \cdot \nabla_{\Gamma} w + \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla_{\Gamma} w \cdot \varepsilon_{\Gamma} + (\nabla w \cdot \nu) (\varepsilon \cdot \nu)) \\ & - \nabla_{\Gamma} u \cdot (\nabla \varepsilon + (\nabla \varepsilon)^T) \cdot \nabla_{\Gamma} w. \end{aligned}$$

In order to handle reasonable expressions we split the computation of the divergence term in proposition 4.4 in two terms that we compute separately.

PROPOSITION 4.6. *We have*

$$\begin{aligned} & (\varepsilon \cdot \nu) \operatorname{div} (u \nabla w(\cdot, x) + \lambda_t u w(\cdot, x) \nu_t) |_{t=0} = \\ & (\varepsilon \cdot \nu) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w(\cdot, x) + M_{\mu} u M_{\mu} w(\cdot, x) - (k^2 + \lambda^2 - \lambda(\operatorname{div}_{\Gamma} \nu) u w(\cdot, x))) \\ & - (\nabla_{\Gamma} \lambda \cdot \varepsilon) u w(\cdot, x), \end{aligned}$$

where we have used the short notation $M_{\mu} \cdot = \operatorname{div}_{\Gamma} (\mu \nabla_{\Gamma} \cdot)$, uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

Proof. We have

$$\operatorname{div} (u \nabla w + \lambda_t u w \nu_t) = \nabla u \cdot \nabla w + u \Delta w + u w (\nabla \lambda_t \cdot \nu_t) + \lambda_t w \nabla u \cdot \nu_t + \lambda_t u \nabla w \cdot \nu_t + \lambda_t u w (\operatorname{div} \nu_t).$$

By using the equation $\Delta w + k^2 w = 0$ and the decomposition of gradient into its normal and tangential parts, we obtain

$$\begin{aligned} \operatorname{div} (u \nabla w + \lambda_t u w \nu_t) |_{t=0} &= \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w + (\nabla u \cdot \nu) (\nabla w \cdot \nu) - (k^2 - \lambda(\operatorname{div} \nu_t) |_{t=0}) u w \\ & + u w (\nabla \lambda_t \cdot \nu_t) |_{t=0} + \lambda (\nabla u \cdot \nu) w + \lambda u (\nabla w \cdot \nu). \end{aligned}$$

We can now replace $\nabla u \cdot \nu$ and $\nabla w \cdot \nu$ by $(-M_{\mu} u - \lambda u)$ and $(-M_{\mu} w - \lambda w)$ respectively, which leads to

$$\begin{aligned} \operatorname{div} (u \nabla w + \lambda_t u w \nu_t) |_{t=0} &= \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w + M_{\mu} u M_{\mu} w - (k^2 + \lambda^2 - \lambda(\operatorname{div} \nu_t) |_{t=0}) u w \\ & + u w (\nabla \lambda_t \cdot \nu_t) |_{t=0}. \end{aligned}$$

We complete the proof by using lemma 4.5. \square

PROPOSITION 4.7. *We have*

$$\begin{aligned} & (\varepsilon \cdot \nu) \operatorname{div} (\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w(\cdot, x) \nu_t) |_{t=0} = \\ & -(\nabla_{\Gamma} \mu \cdot \varepsilon) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w(\cdot, x)) + \mu (\varepsilon \cdot \nu) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w(\cdot, x)) (\operatorname{div}_{\Gamma} \nu) \\ & + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} (\nabla u \cdot \nu) \cdot \nabla_{\Gamma} w(\cdot, x) + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla w \cdot \nu) (\cdot, x) \\ & + \mu (\nabla u \cdot \nu) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} w(\cdot, x) + \mu (\nabla w \cdot \nu) (\cdot, x) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} u \\ & - 2\mu (\varepsilon \cdot \nu) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} \nu \cdot \nabla_{\Gamma} w(\cdot, x)). \end{aligned}$$

Proof. We have

$$\begin{aligned} & \operatorname{div}(\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w(\cdot, x) \nu_t) |_{t=0} \\ &= (\nabla \mu_t \cdot \nu_t) |_{t=0} (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) + \mu (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) (\operatorname{div} \nu_t) |_{t=0} + \mu \nabla (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \cdot \nu_t |_{t=0}. \end{aligned}$$

By using lemma 4.5, we obtain that

$$\begin{aligned} & (\varepsilon \cdot \nu) \operatorname{div}(\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w(\cdot, x) \nu_t) |_{t=0} = -(\nabla_{\Gamma} \mu \cdot \varepsilon) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) \\ & \quad + \mu (\varepsilon \cdot \nu) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) (\operatorname{div} \nu) - \mu \varepsilon_{\Gamma} \cdot \nabla_{\Gamma} (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) \\ & + \mu \nabla_{\Gamma} (\varepsilon_{\Gamma} \cdot \nabla_{\Gamma} u + (\nabla u \cdot \nu) (\varepsilon \cdot \nu)) \cdot \nabla_{\Gamma} w + \mu \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla_{\Gamma} w \cdot \varepsilon_{\Gamma} + (\nabla w \cdot \nu) (\varepsilon \cdot \nu)) \\ & \quad - \mu \nabla_{\Gamma} u \cdot (\nabla \varepsilon + (\nabla \varepsilon)^T) \cdot \nabla_{\Gamma} w. \end{aligned}$$

In the following, for a surface vector a_{Γ} , we denote by $\nabla_{\Gamma} a_{\Gamma}$ the $(d-1) \times (d-1)$ tensor defined by

$$\nabla_{\Gamma} a_{\Gamma} \cdot e_j = \frac{\partial a_{\Gamma}}{\partial \xi_j}, \quad j = 1, \dots, d-1.$$

The third line of the equation above, using $\nabla_{\Gamma} (a_{\Gamma} \cdot b_{\Gamma}) = (\nabla_{\Gamma} a_{\Gamma})^T \cdot b_{\Gamma} + a_{\Gamma} \cdot \nabla_{\Gamma} b_{\Gamma}$, can be expressed as

$$\begin{aligned} & \mu \nabla_{\Gamma} (\varepsilon_{\Gamma} \cdot \nabla_{\Gamma} u + (\nabla u \cdot \nu) (\varepsilon \cdot \nu)) \cdot \nabla_{\Gamma} w + \mu \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla_{\Gamma} w \cdot \varepsilon_{\Gamma} + (\nabla w \cdot \nu) (\varepsilon \cdot \nu)) \\ & = \mu \varepsilon_{\Gamma} \cdot \nabla_{\Gamma} (\nabla_{\Gamma} u) \cdot \nabla_{\Gamma} w + \mu \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varepsilon_{\Gamma} \cdot \nabla_{\Gamma} w \\ & \quad + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} (\nabla u \cdot \nu) \cdot \nabla_{\Gamma} w + \mu (\nabla u \cdot \nu) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} w \\ & \quad + \mu \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla_{\Gamma} w) \cdot \varepsilon_{\Gamma} + \mu \nabla_{\Gamma} u \cdot (\nabla_{\Gamma} \varepsilon_{\Gamma})^T \cdot \nabla_{\Gamma} w \\ & \quad - \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla w \cdot \nu) + \mu (\nabla w \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\varepsilon \cdot \nu). \end{aligned}$$

Gathering the two above expressions and using the fact that

$$\varepsilon_{\Gamma} \cdot \nabla_{\Gamma} (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) = \varepsilon_{\Gamma} \cdot \nabla_{\Gamma} (\nabla_{\Gamma} u) \cdot \nabla_{\Gamma} w + \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla_{\Gamma} w) \cdot \varepsilon_{\Gamma},$$

we obtain that

$$\begin{aligned} & (\varepsilon \cdot \nu) \operatorname{div}(\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w(\cdot, x) \nu_t) |_{t=0} = -(\nabla_{\Gamma} \mu \cdot \varepsilon) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) + \mu (\varepsilon \cdot \nu) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) (\operatorname{div} \nu) \\ & \quad + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} (\nabla u \cdot \nu) \cdot \nabla_{\Gamma} w + \mu (\nabla u \cdot \nu) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} w \\ & \quad + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla w \cdot \nu) + \mu (\nabla w \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\varepsilon \cdot \nu) \\ & \quad - \mu \nabla_{\Gamma} u \cdot (\nabla_{\Gamma} \varepsilon + (\nabla_{\Gamma} \varepsilon)^T) \cdot \nabla_{\Gamma} w + \mu \nabla_{\Gamma} u \cdot (\nabla_{\Gamma} \varepsilon_{\Gamma} + (\nabla_{\Gamma} \varepsilon_{\Gamma})^T) \cdot \nabla_{\Gamma} w. \end{aligned}$$

Now we need to evaluate $\nabla_{\Gamma} \varepsilon - \nabla_{\Gamma} \varepsilon_{\Gamma}$. Since $\varepsilon - \varepsilon_{\Gamma} = (\varepsilon \cdot \nu) \nu$, we have

$$\nabla_{\Gamma} \varepsilon - \nabla_{\Gamma} \varepsilon_{\Gamma} = \nabla_{\Gamma} ((\varepsilon \cdot \nu) \nu) = (\varepsilon \cdot \nu) \nabla_{\Gamma} \nu + \nu \otimes \nabla_{\Gamma} (\varepsilon \cdot \nu),$$

where for a surface field a , we denote by $\nu \otimes \nabla_{\Gamma} a$ the $d \times d$ tensor M defined by

$$M \cdot e_j = \frac{\partial a}{\partial \xi_j} \nu \quad j = 1, \dots, d-1, \quad M \cdot \nu = 0.$$

This implies that

$$\nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varepsilon_{\Gamma} \cdot \nabla_{\Gamma} w - \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varepsilon \cdot \nabla_{\Gamma} w = -(\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \nu \cdot \nabla_{\Gamma} w.$$

Since the tensor $\nabla_{\Gamma} \nu$ is symmetric (see for example [22, theorem 2.5.18]), we also obtain

$$\nabla_{\Gamma} u \cdot (\nabla_{\Gamma} \varepsilon_{\Gamma})^T \cdot \nabla_{\Gamma} w - \nabla_{\Gamma} u \cdot (\nabla_{\Gamma} \varepsilon)^T \cdot \nabla_{\Gamma} w = -(\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \nu \cdot \nabla_{\Gamma} w.$$

We finally arrive at

$$\begin{aligned}
& (\varepsilon \cdot \nu) \operatorname{div} (\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w(\cdot, x) \nu_t) |_{t=0} \\
&= -(\nabla_{\Gamma} \mu \cdot \varepsilon) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) + \mu (\varepsilon \cdot \nu) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) (\operatorname{div}_{\Gamma} \nu) \\
&\quad + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} (\nabla u \cdot \nu) \cdot \nabla_{\Gamma} w + \mu (\nabla u \cdot \nu) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} w \\
&\quad + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla w \cdot \nu) + \mu (\nabla w \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\varepsilon \cdot \nu) - 2\mu (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \nu \cdot \nabla_{\Gamma} w,
\end{aligned}$$

which completes the proof. \square

Gathering propositions 4.6 and 4.7, we establish the main theorem of this section, that is

THEOREM 4.8. *The discrepancy between the scattered fields due to obstacle D_ε and the obstacle D is*

$$u_\varepsilon^s(x) - u^s(x) = - \int_{\partial D} B_\varepsilon u(y) w(y, x) ds(y) + \mathcal{O}(\|\varepsilon\|^2),$$

uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$, where $w(\cdot, x)$ is the solution of problem (2.1) associated with $u^i = \Phi(\cdot, x)$, and the surface operator B_ε is defined by

$$\begin{aligned}
B_\varepsilon u &= (\varepsilon \cdot \nu) (k^2 - 2H\lambda) u + \operatorname{div}_{\Gamma} ((Id + 2\mu(R - H Id)) (\varepsilon \cdot \nu) \nabla_{\Gamma} u) + L_{\lambda, \mu} ((\varepsilon \cdot \nu) L_{\lambda, \mu} u) \\
&\quad + (\nabla_{\Gamma} \lambda \cdot \varepsilon_{\Gamma}) u + \operatorname{div}_{\Gamma} ((\nabla_{\Gamma} \mu \cdot \varepsilon_{\Gamma}) \nabla_{\Gamma} u),
\end{aligned}$$

with $2H := \operatorname{div}_{\Gamma} \nu$, $R := \nabla_{\Gamma} \nu$ and $L_{\lambda, \mu} \cdot := \operatorname{div}_{\Gamma} (\mu \nabla_{\Gamma} \cdot) + \lambda \cdot$.

Proof. From propositions 4.6 and 4.7 it follows that

$$\begin{aligned}
& (\varepsilon \cdot \nu) \operatorname{div} (-\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w \nu_t + u \nabla w + \lambda_t u w \nu_t) |_{t=0} \\
&= (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w + (\varepsilon \cdot \nu) M_{\mu} u M_{\mu} w - (\varepsilon \cdot \nu) (k^2 + \lambda^2 - \lambda (\operatorname{div}_{\Gamma} \nu)) u w \\
&\quad - (\nabla_{\Gamma} \lambda \cdot \varepsilon) u w + (\nabla_{\Gamma} \mu \cdot \varepsilon) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w - \mu (\operatorname{div}_{\Gamma} \nu) (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w \\
&\quad - \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} (\nabla u \cdot \nu) \cdot \nabla_{\Gamma} w - \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (\nabla w \cdot \nu) \\
&\quad - \mu (\nabla u \cdot \nu) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} w - \mu (\nabla w \cdot \nu) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} u \\
&\quad + 2\mu (\varepsilon \cdot \nu) (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} \nu \cdot \nabla_{\Gamma} w).
\end{aligned}$$

Using the boundary condition for u and $w(\cdot, x)$ on ∂D , we obtain

$$\begin{aligned}
& (\varepsilon \cdot \nu) \operatorname{div} (-\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w \nu_t + u \nabla w + \lambda_t u w \nu_t) |_{t=0} \\
&= -(\varepsilon \cdot \nu) (k^2 - 2H\lambda) u w + (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot (Id + 2\mu(R - H Id)) \cdot \nabla_{\Gamma} w \\
&\quad - (\nabla_{\Gamma} \lambda \cdot \varepsilon) u w + (\nabla_{\Gamma} \mu \cdot \varepsilon) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} w \\
&\quad + (\varepsilon \cdot \nu) M_{\mu} u M_{\mu} w - (\varepsilon \cdot \nu) \lambda^2 u w \\
&\quad + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} (M_{\mu} u + \lambda u) \cdot \nabla_{\Gamma} w + \mu (\varepsilon \cdot \nu) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} (M_{\mu} w + \lambda w) \\
&\quad + \mu (M_{\mu} u + \lambda u) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} w + \mu (M_{\mu} w + \lambda w) \nabla_{\Gamma} (\varepsilon \cdot \nu) \cdot \nabla_{\Gamma} u.
\end{aligned}$$

The three last lines of the above expression may be written as

$$\begin{aligned}
& (\varepsilon \cdot \nu) (L_{\lambda, \mu} u) (L_{\lambda, \mu} w) - \lambda (\varepsilon \cdot \nu) w (L_{\lambda, \mu} u) - \lambda (\varepsilon \cdot \nu) u (L_{\lambda, \mu} w) \\
&\quad + \mu \nabla_{\Gamma} ((\varepsilon \cdot \nu) L_{\lambda, \mu} u) \cdot \nabla_{\Gamma} w + \mu \nabla_{\Gamma} ((\varepsilon \cdot \nu) L_{\lambda, \mu} w) \cdot \nabla_{\Gamma} u.
\end{aligned}$$

The integral over ∂D of the above expression is, after integration by parts and simplification,

$$- \int_{\partial D} (\varepsilon \cdot \nu) (L_{\lambda, \mu} u) (L_{\lambda, \mu} w) ds.$$

To complete the proof, we simply use proposition 4.4 and integration by parts. \square

COROLLARY 4.9. *We assume that ∂D , λ and μ are analytic, and (λ, μ) satisfy assumption 1. Then the farfield operator $T : (\lambda, \mu, \partial D) \rightarrow u^\infty$ is differentiable with respect to ∂D according to definition 4.1 and its Fréchet derivative is given by*

$$T'_{\lambda, \mu}(\partial D) \cdot \varepsilon = v_\varepsilon^\infty,$$

where v_ε^∞ is the farfield associated with the outgoing solution v_ε^s of the Helmholtz equation outside D which satisfies the GIBC condition

$$\frac{\partial v_\varepsilon^s}{\partial \nu} + \operatorname{div}(\mu \nabla_\Gamma v_\varepsilon^s) + \lambda v_\varepsilon^s = B_\varepsilon u \quad \text{on } \partial D,$$

where $B_\varepsilon u$ is given by theorem 4.8.

Proof. Proceeding as in [14], we can drop the assumption $\overline{D} \subset D_\varepsilon$ provided we assume that ∂D , λ and μ be analytic. The result then follows from theorem 4.8 and an integral representation for the scattered field $v_\varepsilon^s = u_\varepsilon^s - u^s$. \square

REMARK 2. *With classical impedance boundary condition, that is $\mu = 0$, we retrieve the result of [14, theorem 2.5]. Let us also remark that in this case the surface operator B_ε in theorem 4.8 is a second-order operator, while it becomes a fourth-order differential operator when $\mu \neq 0$.*

REMARK 3. *Classically, the shape derivative only involves the normal part $(\varepsilon \cdot \nu)$ of field ε (see for example [16, proposition 5.9.1]). In view of theorem 4.8, the expression of B_ε may be split in two parts: one part involves the normal component $(\varepsilon \cdot \nu)$, the second part involves the tangential component ε_Γ . This is due to the fact that the impedances λ and μ are surface functions.*

5. An optimization technique to solve the inverse problem. This section is dedicated to the effective reconstruction of both the obstacle ∂D and the functional impedances (λ, μ) from the observed farfields $u_{\text{obs},j}^\infty := T_j(\lambda_0, \mu_0, \partial D_0) \in L^2(S^{d-1})$ associated with N given plane wave directions, where T_j refers to incident direction \hat{d}_j . We shall minimize the cost function

$$F(\lambda, \mu, \partial D) = \frac{1}{2} \sum_{j=1}^N \|T_j(\lambda, \mu, \partial D) - u_{\text{obs},j}^\infty\|_{L^2(S^{d-1})}^2 \quad (5.1)$$

with respect to ∂D and (λ, μ) by using a steepest descent method.

To do so, we first compute the Fréchet derivative of T with respect to (λ, μ) for fixed D . We have the following theorem.

THEOREM 5.1. *We assume that D is Lipschitz continuous. Then for $(\lambda, \mu) \in (L^\infty(\partial D))^2$ which satisfy assumption 1, the function $T : (\lambda, \mu, \partial D) \rightarrow u^\infty$ is Fréchet differentiable with respect to (λ, μ) and its Fréchet derivative is given by*

$$T'_{\partial D}(\lambda, \mu) \cdot (h, l) = v_{h,l}^\infty(\hat{x}) := \langle p(\cdot, \hat{x}), \operatorname{div}_\Gamma(l \nabla_\Gamma u) + h u \rangle_{H^1(\Gamma), H^{-1}(\Gamma)}, \quad \forall \hat{x} \in S^{d-1},$$

where

- u is the solution of the problem (2.1),
- $p(\cdot, \hat{x}) = \Phi^\infty(\cdot, \hat{x}) + p^s(\cdot, \hat{x})$ is the solution of (2.1) with u^i replaced by $\Phi^\infty(\cdot, \hat{x})$.

Proof. The proof of this result can be found in [7]. \square

The Fréchet derivative of T with respect to ∂D for fixed (λ, μ) is given by theorem 4.8 and its corollary 4.9. With the help of corollary 4.9 and theorem 5.1, and in the case ∂D , λ and μ are analytic, we obtain the following expressions for the partial derivatives of the cost function F with respect to (λ, μ) and ∂D respectively.

$$F'_{\partial D}(\lambda, \mu) \cdot (h, l) = \sum_{j=1}^N \Re e \left(\int_{\partial D} G_j(\operatorname{div}_\Gamma(l \nabla_\Gamma u_j) + h u_j) dy \right), \quad (5.2)$$

$$F'_{\lambda, \mu}(\partial D) \cdot \varepsilon = - \sum_{j=1}^N \Re e \left(\int_{\partial D} G_j(B_\varepsilon u_j) dy \right) \quad (5.3)$$

where

- u_j is the solution of the problem (2.1) wich is associated to plane wave direction \hat{d}_j ,
- $G_j = G_j^i + G_j^s$ is the solution of problem (2.1) with u^i replaced by

$$G_j^i(y) := \int_{S^{d-1}} \Phi^\infty(y, \hat{x}) \overline{(T_j(\lambda, \mu, \partial D) - u_{\text{obs},j}^\infty)} d\hat{x}.$$

In the numerical part of the paper we restrict ourselves to the two dimensional setting, that is $d = 2$. The minimization of the cost function F alternatively with respect to D , λ and μ relies on the directions of steepest descent given by (5.2) and (5.3). The minimization with respect to (λ, μ) is already exposed in [7], so that we only describe the minimization with respect to D . It is essential to remark from theorem 4.8 that the partial derivative with respect to D depends only on the values of ε on ∂D . With the decomposition $\varepsilon = \varepsilon_\tau \tau + \varepsilon_\nu \nu$, where τ is the tangential unit vector, we formally compute $(\varepsilon_\tau, \varepsilon_\nu)$ on ∂D such that

$$\varepsilon_\tau \tau + \varepsilon_\nu \nu = -\alpha F'_{\lambda, \mu}(\partial D),$$

where $\alpha > 0$ is the descent coefficient. In order to decrease the oscillations of the updated boundary, similarly to [7] we use a H^1 -regularization, that is we search ε_τ and ε_ν in $H^1(\partial D)$ such that for all $\phi \in H^1(\partial D)$,

$$\eta_\tau \int_{\partial D} \nabla_\Gamma \varepsilon_\tau \cdot \nabla_\Gamma \phi \, ds + \int_{\partial D} \varepsilon_\tau \phi \, ds = -\alpha F'_{\lambda, \mu}(\partial D) \cdot (\phi \tau), \quad (5.4)$$

$$\eta_\nu \int_{\partial D} \nabla_\Gamma \varepsilon_\nu \cdot \nabla_\Gamma \phi \, ds + \int_{\partial D} \varepsilon_\nu \phi \, ds = -\alpha F'_{\lambda, \mu}(\partial D) \cdot (\phi \nu), \quad (5.5)$$

where $\eta_\tau, \eta_\nu > 0$ are regularization coefficients, while $F'_{\lambda, \mu}(\partial D) \cdot (\phi \tau)$ and $F'_{\lambda, \mu}(\partial D) \cdot (\phi \nu)$ are given by (5.3) (see more explicit expressions in [6]). The updated obstacle D_ε is then obtained by moving the mesh points x of ∂D to the points x_ε defined by $x_\varepsilon = x + (\varepsilon_\tau \tau + \varepsilon_\nu \nu)(x)$, while the extended impedances on ∂D_ε are defined, following (4.1), by $\lambda_\varepsilon(x_\varepsilon) = \lambda(x)$ and $\mu_\varepsilon(x_\varepsilon) = \mu(x)$. The points x_ε enable us to define a new domain D_ε , and we have to remesh the complementary domain $\Omega_R^\varepsilon = B_R \setminus \overline{D_\varepsilon}$ to solve the next forward problems. The descent coefficient α and the regularization parameters η_τ, η_ν are determined as follows: α is increased (resp. decreased) and η_τ, η_ν are decreased (resp. increased) as soon as the cost function decreases (resp. increases). The algorithm stops as soon as α is too small. With the help of the relative cost function, namely

$$\text{Error} := \frac{1}{N} \sum_{j=1}^N \frac{\|T_j(\lambda, \mu, \partial D) - u_{\text{obs}, j}^\infty\|_{L^2(S^1)}}{\|u_{\text{obs}, j}^\infty\|_{L^2(S^1)}}, \quad (5.6)$$

we are able to determine if the computed $(\lambda, \mu, \partial D)$ corresponds to a global or a local minimum: in the first case Error is approximately equal to the amplitude of noise while in the second case it is much larger.

6. Some numerical results. In order to handle dimensionless impedances, we replace λ by $k\lambda$ and μ by μ/k in the boundary condition of problem (2.3) without changing the notations. Problem (2.3) is solved by using a finite element method based on the variational formulation associated with problem (2.3) and which is introduced in [5], more precisely we have used classical Lagrange finite elements. The variational formulations (5.4) (5.5) as well as those used to update the impedances λ and μ (see [5]) are solved by using the same finite element basis. All computations were performed with the help of the software FreeFem++ [27]. We obtain some artificial data with forward computations for some given data $(\lambda_0, \mu_0, \partial D_0)$. The resulting farfields $u_{\text{obs}, j}^\infty$, $j = 1, N$ are then contaminated by some Gaussian noise of various amplitude. More precisely, for each Fourier coefficient of the farfield we compute a Gaussian noise with normal distribution. Such a perturbation is multiplied by a constant which is calibrated in order to obtain a global relative L^2 error of prescribed amplitude: 1% or 5%.

6.1. Reconstruction of an obstacle with known impedances. First we try to reconstruct a L -shaped obstacle D_0 with known impedances (λ_0, μ_0) . The result is shown on figure 6.1 in the case of 1% noise by using only two incident waves, namely $N = 2$. The results are shown on figure 6.2 in the case of 5% noise and $N = 2$, as well as 5% noise and $N = 8$, respectively. This enables us to test the influence of the amplitude of noise as well as the influence of the number of incident waves. In order to evaluate the impact of the initial guess on the quality of

the reconstruction, we consider another initial guess which is farther from the true obstacle than the first one, in the presence of 5% noise. The result is very bad with only two incident waves, and becomes much better with eight incident waves, as shown on figure 6.3. In the remainder of the numerical section all reconstructions will be based on eight incident waves.

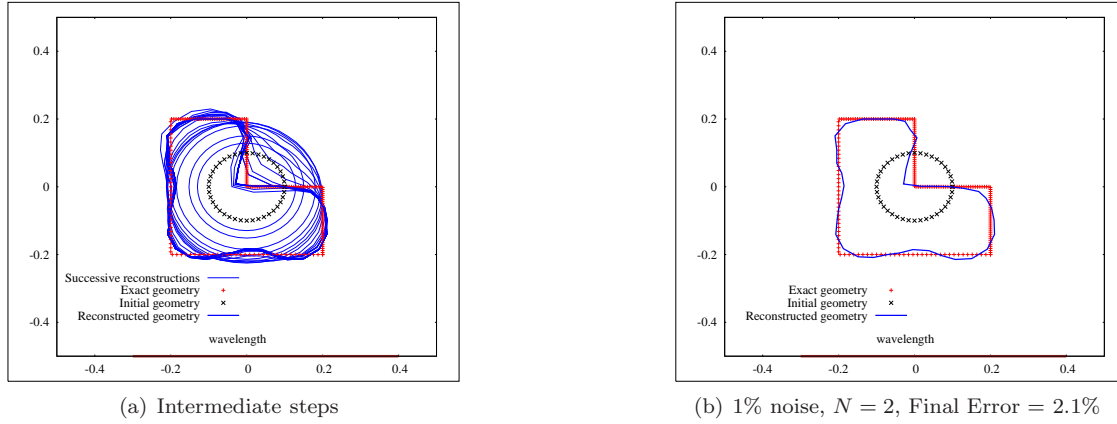


FIG. 6.1. Case of known impedances and good initial guess

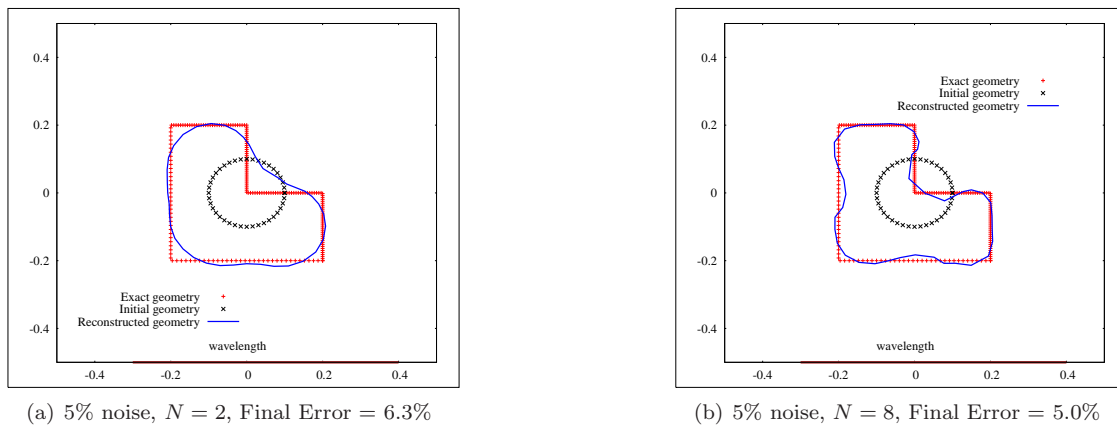


FIG. 6.2. Case of known impedances and good initial guess, influence of noise and of N

6.2. Reconstruction of the geometry and constant impedances. Secondly we assume that both the obstacle D_0 and the impedances $(\lambda_0, \mu_0) = (0.5i, 2)$ are unknown, but these impedances are constants. Starting from $(i, 1.5)$ as initial guess for (λ, μ) , the retrieved impedances are $(\lambda, \mu) = (0.49i, 1.99)$ for 1% noise and $(\lambda, \mu) = (0.51i, 1.93)$ for 5% noise, while the corresponding retrieved obstacles are shown on figure 6.4.

6.3. Reconstruction of the geometry and functional impedances. In order to emphasize the role played by the tangential part of the mapping ε in the optimization of the cost function F for functional impedances (see remark 3), we first consider a very academic case. We try to reconstruct a circle D_0 of radius $R_0 = 0.3$ and an impedance $\lambda_0(\theta) = 0.5(1 + \sin^2(\theta + \frac{\pi}{6}))$, where θ is the polar angle, starting from an initial circle of same center and radius 0.2 and from the initial impedance $\lambda(\theta) = 0.5(1 + \sin^2(\theta))$. Compared to the true obstacle, the initial guess is hence a smaller and rotated circle. Here $\mu = 0$ for sake of simplicity. Amplitude of noise is 5% and we use eight incident waves. As can be seen on figure 6.5, the obstacle D_0 and the impedance λ_0 are quite well reconstructed even if we use only the gradient iterations on the geometry (we do

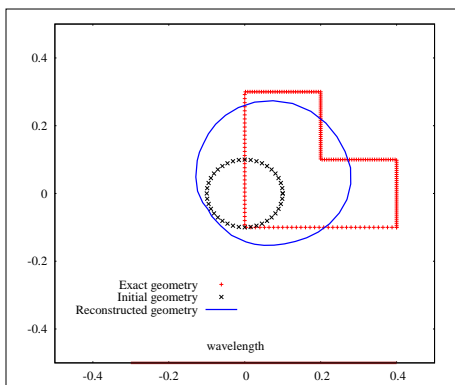
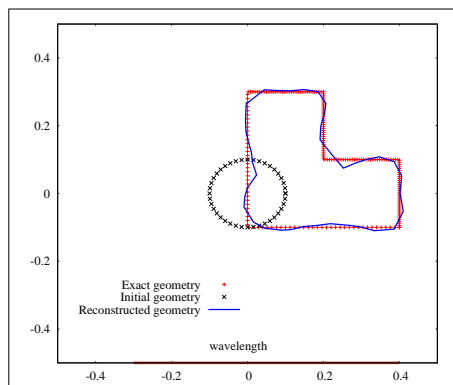
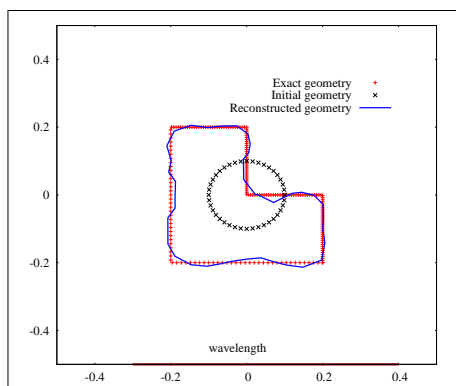
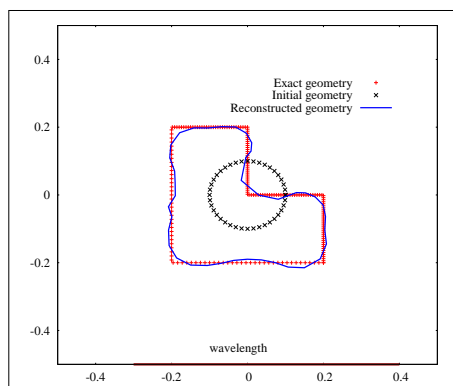
(a) 5% noise, $N = 2$, Final Error = 25.2%(b) 5% noise, $N = 8$, Final Error = 5.8%

FIG. 6.3. Case of known impedances and bad initial guess, increasing number of incident waves



(a) 1% of noise, Final Error = 1.4%



(b) 5% of noise, Final Error = 5.8%

FIG. 6.4. Case of constant but unknown impedances

not use the gradient iterations on the impedance).

We complete the numerical section with a more complicated example. The aim is to retrieve the obstacle D_0 defined with polar coordinates (r, θ) by $r = 0.3 + 0.08 \cos(3\theta)$, as well as the impedances $\lambda_0 = 0.5(1 + \sin^2 \theta)i$ and $\mu_0 = 0.5(1 + \cos^2 \theta)$, assuming that both the real part of λ and the imaginary part of μ are 0, in the presence of 5% noise with eight incident waves. Note that in this case the obstacle is star-shaped, which is not necessary to apply our optimization process, but it enables us to compare the retrieved and the exact impedances in a simple way. The result are presented on figure 6.6 and show a good accuracy.

Appendix. We give below the proof of lemma 4.5. In order to obtain this lemma, we consider the local basis (e_j^t, e_d^t) , $j = 1, d-1$, where vectors e_j^t are defined by (4.4), while $e_d^t = \varepsilon$. We can hence define the associated covariant basis (f_t^i) , $i = 1, d$. Note that $f_t^i \neq e_t^i$ ($i = 1, d-1$), where covariant vectors e_t^i are defined by (4.5). We begin with the proof of the first part of lemma 4.5. We have, denoting $\tilde{\lambda} = \lambda \circ \phi$

$$\nabla \lambda_t = \sum_{i=1}^{d-1} \frac{\partial \tilde{\lambda}}{\partial \xi_i} f_t^i + \frac{\partial \tilde{\lambda}}{\partial t} f_t^d.$$

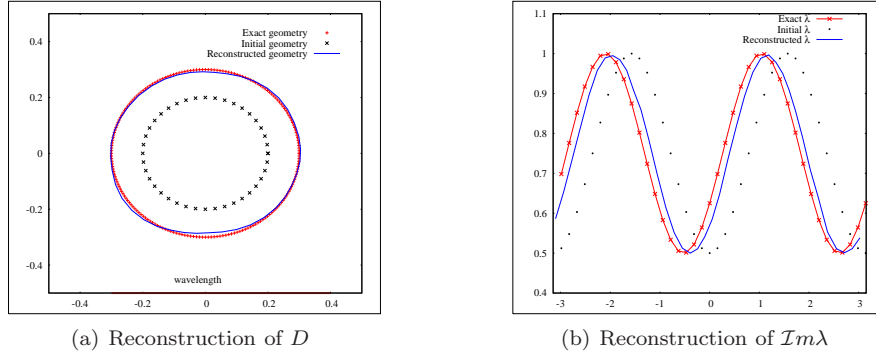
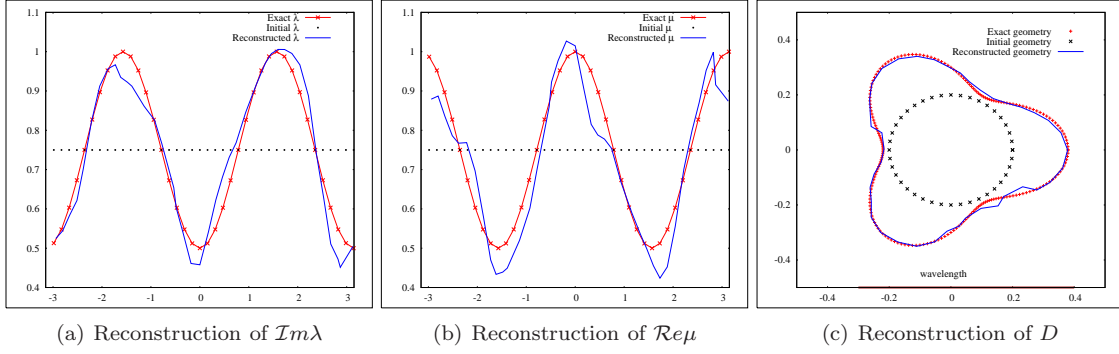
FIG. 6.5. On the use of the tangential component of mapping ε 

FIG. 6.6. Case of unknown geometry and unknown functional impedances

By the definition of $\tilde{\lambda}$, we have $\partial\tilde{\lambda}/\partial t = 0$. We hence have, with $f^i := f_0^i$,

$$(\nabla\lambda_t)|_{t=0} = \sum_{i=1}^{d-1} \frac{\partial\tilde{\lambda}}{\partial\xi_i} f^i. \quad (6.1)$$

It remains to compute the covariant vectors f^i for $i = 1, d-1$. In this view we search f^1 in the form

$$f^1 = \sum_{i=1}^{d-1} \beta_i e^i + \alpha\nu.$$

The coefficients α, β_i are uniquely defined by

$$f^1 \cdot e_1 = 1, \quad f^1 \cdot e_j = 0, \quad j = 2, d-1, \quad f^1 \cdot \varepsilon = 0.$$

This implies that

$$\beta_1 = 1, \quad \beta_j = 0, \quad j = 2, d-1, \quad \alpha(\nu \cdot \varepsilon) = -e^1 \cdot \varepsilon.$$

As a conclusion, we have

$$f^1 = e^1 - \frac{1}{\nu \cdot \varepsilon} (e^1 \cdot \varepsilon) \nu.$$

We obtain a symmetric expression for f^i , $i = 2, d-1$ and coming back to (6.1), we obtain

$$(\nabla\lambda_t \cdot \nu_t)|_{t=0} = (\nabla\lambda_t)|_{t=0} \cdot \nu = - \sum_{i=1}^{d-1} \frac{1}{\nu \cdot \varepsilon} (e^i \cdot \varepsilon) \frac{\partial\tilde{\lambda}}{\partial\xi_i},$$

and lastly,

$$(\nu \cdot \varepsilon)(\nabla \lambda_t \cdot \nu_t)|_{t=0} = -(\nabla_{\Gamma} \lambda \cdot \varepsilon),$$

which completes the proof of the first statement of lemma 4.5.

Now let us give the proof of the second statement of lemma 4.5. In this view we also need an expression of the covariant vector f^d . We again search f^d in the form

$$f^d = \sum_{i=1}^{d-1} \beta_i e^i + \alpha \nu.$$

The coefficients α, β_i are now uniquely defined by

$$f^d \cdot e_i = 0, \quad i = 1, d-1, \quad f^d \cdot \varepsilon = 1.$$

After simple calculations, we obtain

$$f^d = \frac{1}{\nu \cdot \varepsilon} \nu.$$

We have

$$\operatorname{div} \nu_t = \sum_{i=1}^{d-1} \frac{\partial \nu_t}{\partial \xi_i} \cdot f_t^i + \frac{\partial \nu_t}{\partial t} \cdot f_t^d.$$

By differentiation of $|\nu_t|^2 = 1$ with respect to ξ_i and t , we obtain

$$\frac{\partial \nu_t}{\partial \xi_i} \Big|_{t=0} \cdot \nu = 0, \quad i = 1, d-1, \quad \frac{\partial \nu_t}{\partial t} \Big|_{t=0} \cdot \nu = 0,$$

hence

$$\left(\sum_{i=1}^{d-1} \frac{\partial \nu_t}{\partial \xi_i} \cdot f_t^i \right) \Big|_{t=0} = \sum_{i=1}^{d-1} \frac{\partial \nu}{\partial \xi_i} \cdot e^i = \operatorname{div}_{\Gamma} \nu,$$

and we obtain the second thesis of lemma 4.5.

Lastly, let us give the proof of the third statement of lemma 4.5. Let us denote

$$G = \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w$$

and $\tilde{G}_t = G \circ \phi_t$. Given the definition of surface gradient (4.6), we have

$$\nabla G = \sum_{i=1}^{d-1} \frac{\partial \tilde{G}_t}{\partial \xi_i} f_t^i + \frac{\partial \tilde{G}_t}{\partial t} f_t^d.$$

By using the expressions obtained above for the covariant vectors f_t^i , $i = 1, d$, we obtain

$$(\nu \cdot \varepsilon)(\nabla G \cdot \nu_t) \Big|_{t=0} = - \sum_{i=1}^{d-1} (e^i \cdot \varepsilon) \frac{\partial \tilde{G}_t}{\partial \xi_i} \Big|_{t=0} + \frac{\partial \tilde{G}_t}{\partial t} \Big|_{t=0},$$

that is

$$(\nu \cdot \varepsilon)(\nabla G \cdot \nu_t) \Big|_{t=0} = -\varepsilon_{\Gamma} \cdot \nabla_{\Gamma} (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} w) + \frac{\partial \tilde{G}_t}{\partial t} \Big|_{t=0}. \quad (6.2)$$

We now have to compute $\partial \tilde{G}_t / \partial t$ at $t = 0$. We have

$$\frac{\partial \tilde{G}_t}{\partial t} = \sum_{i,j=1}^{d-1} \frac{\partial}{\partial t} \left(\frac{\partial \tilde{u}_t}{\partial \xi_i} e_t^i \cdot \frac{\partial \tilde{w}_t}{\partial \xi_j} e_t^j \right),$$

with

$$\frac{\partial}{\partial t} \left(\frac{\partial \tilde{u}_t}{\partial \xi_i} e_t^i \cdot \frac{\partial \tilde{w}_t}{\partial \xi_j} e_t^j \right) = \frac{\partial^2 \tilde{u}_t}{\partial \xi_i \partial t} \frac{\partial \tilde{w}_t}{\partial \xi_j} e_t^i \cdot e_t^j + \frac{\partial \tilde{u}_t}{\partial \xi_i} \frac{\partial^2 \tilde{w}_t}{\partial \xi_j \partial t} e_t^i \cdot e_t^j + \frac{\partial \tilde{u}_t}{\partial \xi_i} \frac{\partial \tilde{w}_t}{\partial \xi_j} \left(\frac{\partial e_t^i}{\partial t} \cdot e_t^j + e_t^i \cdot \frac{\partial e_t^j}{\partial t} \right).$$

From differentiation with respect to t of

$$(Id + t(\nabla \varepsilon)^T) e_t^i = e^i,$$

we obtain

$$(\nabla \varepsilon)^T e_t^i + (Id + t(\nabla \varepsilon)^T) \frac{\partial e_t^i}{\partial t} = 0,$$

hence

$$\frac{\partial e_t^i}{\partial t} = -(Id + t(\nabla \varepsilon)^T)^{-1} (\nabla \varepsilon)^T e_t^i,$$

and in particular

$$\frac{\partial e_t^i}{\partial t} \Big|_{t=0} = -(\nabla \varepsilon)^T \cdot e^i.$$

We arrive at

$$\begin{aligned} \frac{\partial \tilde{G}_t}{\partial t} \Big|_{t=0} &= \nabla_\Gamma \left(\frac{\partial \tilde{u}_t}{\partial t} \Big|_{t=0} \right) \cdot \nabla_\Gamma w + \nabla_\Gamma u \cdot \nabla_\Gamma \left(\frac{\partial \tilde{w}_t}{\partial t} \Big|_{t=0} \right) \\ &\quad - \nabla_\Gamma u \cdot \nabla \varepsilon \cdot \nabla_\Gamma w - \nabla_\Gamma u \cdot (\nabla \varepsilon)^T \cdot \nabla_\Gamma w. \end{aligned}$$

By using

$$\frac{\partial \tilde{u}_t}{\partial t} \Big|_{t=0} = \nabla u \cdot \varepsilon, \quad \frac{\partial \tilde{w}_t}{\partial t} \Big|_{t=0} = \nabla w \cdot \varepsilon$$

as well as the decomposition $\varepsilon = \varepsilon_\Gamma + (\varepsilon \cdot \nu)\nu$, we obtain

$$\begin{aligned} \frac{\partial \tilde{G}_t}{\partial t} \Big|_{t=0} &= \nabla_\Gamma (\nabla_\Gamma u \cdot \varepsilon_\Gamma + (\nabla u \cdot \nu)(\varepsilon \cdot \nu)) \cdot \nabla_\Gamma w + \nabla_\Gamma u \cdot \nabla_\Gamma (\nabla_\Gamma w \cdot \varepsilon_\Gamma + (\nabla w \cdot \nu)(\varepsilon \cdot \nu)) \\ &\quad - \nabla_\Gamma u \cdot (\nabla \varepsilon + (\nabla \varepsilon)^T) \cdot \nabla_\Gamma w. \end{aligned}$$

We complete the proof of lemma 4.5 by using equation (6.2).

Acknowledgments. The work of Nicolas Chaulet is supported by a grant from Délégation Générale de l'Armement.

REFERENCES

- [1] G. ALLAIRE, *Conception optimale de structures*, Springer-Verlag, 2007.
- [2] B. ASLANYREK, H. HADDAR, AND H. SAHINTRK, *Generalized impedance boundary conditions for thin dielectric coatings with variable thickness*, Wave Motion, 48 (2011), pp. 680 – 699.
- [3] V. BACCHELLI, *Uniqueness for the determination of unknown boundary and impedance with the homogeneous Robin condition*, Inverse Problems, 25 (2009), pp. 015004, 4.
- [4] A. BENDALI AND K. LEMRABET, *The effect of a thin coating on the scattering of a time-harmonic wave for the helmholtz equation*, SIAM J. Appl. Math., 56 (1996), pp. 1664–1693.
- [5] L. BOURGEOIS, N. CHAULET, AND H. HADDAR, *Identification of generalized impedance boundary conditions: some numerical issues*, Tech. Report INRIA 7449, 11 2010.
- [6] ———, *On simultaneous identification of a scatterer and its generalized impedance boundary condition*, Tech. Report INRIA 7645, 6 2011.
- [7] ———, *Stable reconstruction of generalized impedance boundary conditions*, Inverse Problems, 27 (2011), p. 095002.

- [8] L. BOURGEOIS AND H. HADDAR, *Identification of generalized impedance boundary conditions in inverse scattering problems*, Inverse Probl. Imaging, 4 (2010), pp. 19–38.
- [9] F. CAKONI, R. KRESS, AND C. SCHUFT, *Integral equations for shape and impedance reconstruction in corrosion detection*, Inverse Problems, 26 (2010), pp. 095012, 24.
- [10] D. COLTON AND R. KRESS, *Inverse acoustic and electromagnetic scattering theory*, vol. 93 of Applied Mathematical Sciences, Springer-Verlag, Berlin, second ed., 1998.
- [11] M. DURUFLÉ, H. HADDAR, AND J. JOLY, *Higher order generalized impedance boundary conditions in electromagnetic scattering problems*, C.R. Physique, 7 (2006), pp. 533–542.
- [12] H. HADDAR AND P. JOLY, *Stability of thin layer approximation of electromagnetic waves scattering by linear and nonlinear coatings*, J. Comput. Appl. Math., 143 (2002), pp. 201–236.
- [13] H. HADDAR, P. JOLY, AND H.-M. NGUYEN, *Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case*, Math. Models Methods Appl. Sci., 15 (2005), pp. 1273–1300.
- [14] H. HADDAR AND R. KRESS, *On the Fréchet derivative for obstacle scattering with an impedance boundary condition*, SIAM J. Appl. Math., 65 (2004), pp. 194–208 (electronic).
- [15] L. HE, S. KINDERMANN, AND M. SINI, *Reconstruction of shapes and impedance functions using few far-field measurements*, Journal of Computational Physics, 228 (2009), pp. 717–730.
- [16] A. HENROT AND M. PIERRE, *Variation et optimisation de formes*, vol. 48 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].
- [17] V. ISAKOV, *On uniqueness in the inverse transmission scattering problem*, Comm. Partial Differential Equations, 15 (1990), pp. 1565–1587.
- [18] A. KIRSCH AND R. KRESS, *Uniqueness in inverse obstacle scattering*, Inverse Problems, 9 (1993), pp. 285–299.
- [19] R. KRESS AND L. PÄIVÄRINTA, *On the far field in obstacle scattering*, SIAM J. Appl. Math., 59 (1999).
- [20] R. KRESS AND W. RUNDELL, *Inverse scattering for shape and impedance*, Inverse Problems, 17 (2001), p. 1075.
- [21] J. J. LIU, G. NAKAMURA, AND M. SINI, *Reconstruction of the shape and surface impedance from acoustic scattering data for an arbitrary cylinder*, SIAM J. Appl. Math., 67 (2007), pp. 1124–1146.
- [22] J.-C. NÉDÉLEC, *Acoustic and electromagnetic equations*, Applied Mathematical Sciences, Springer-Verlag, Berlin, 2001.
- [23] W. RUNDELL, *Recovering an obstacle and a nonlinear conductivity from Cauchy data*, Inverse Problems, 24 (2008), pp. 055015, 12.
- [24] P. SERRANHO, *A hybrid method for inverse scattering for shape and impedance*, Inverse Problems, 22 (2006), pp. 663–680.
- [25] E. SINCICH, *Stability for the determination of unknown boundary and impedance with a robin boundary condition*, SIAM J. Math. Anal., 42 (2010), pp. 2922–2943.
- [26] J. SOKOŁOWSKI AND J.-P. ZOLÉSIO, *Introduction to shape optimization*, vol. 16 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1992. Shape sensitivity analysis.
- [27] WWW.FREEFEM.ORG/FF++, *FreeFem++*.