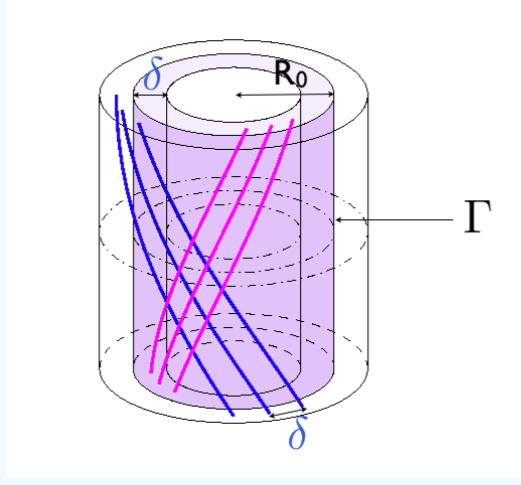


## Context



- Thickness of the ring  $\delta$ , Angular periodicity  $\approx \delta$ .
- $\lambda = \frac{2\pi}{\omega} \gg \delta, \delta \ll R_0$ .
- **Difficulty**: two different scales  $\delta, \lambda$ .

**Goal**: replacing the periodic ring by an approximate transmission condition across  $\Gamma$ .

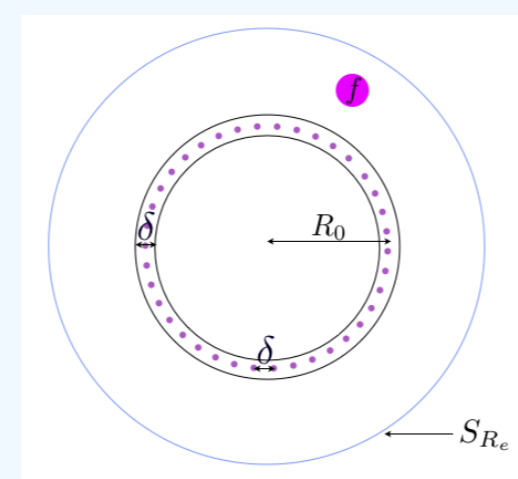
**Method**:

- **Asymptotic expansion** of the solution with respect to the small parameter  $\delta$ : matched asymptotic expansion / homogenisation (see [1], [2],[3]).
- **Construction of stable approximate models** using this expansion.

## Two Dimensional Model Problem

### Description

$$\begin{cases} \nabla \cdot (\mu^\delta \nabla u^\delta) + \omega^2 \rho^\delta u^\delta = f, \\ \partial_r u^\delta + i\omega u^\delta = 0 \text{ on } S_{R_c} \end{cases} \quad \lambda = \frac{2\pi}{\omega} \gg \delta, \quad R_0 \gg \delta$$

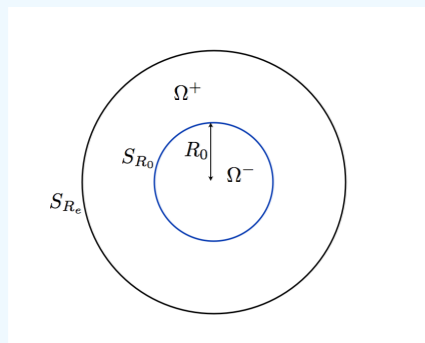


- $(\rho^\delta, \mu^\delta) = (\rho_\infty, \mu_\infty)$  outside the periodic ring,  $(\rho^\delta, \mu^\delta)$  periodic in  $\theta$  of periodicity  $\frac{\delta}{R_0}$
- Thickness of the ring  $\delta$  ( $\approx$  periodicity of the ring)

### Asymptotic Expansion

Far field:

$$(1) \quad u^\delta = \sum_{n \in \mathbb{N}} \delta^n u_n(r, \theta)$$

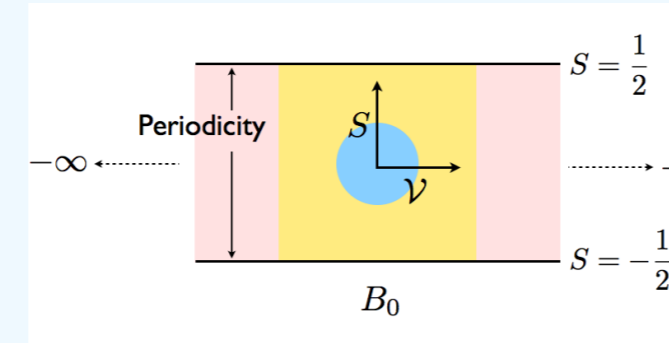


$$\mu_\infty \Delta u_n^\pm + \omega^2 \rho_\infty u_n^\pm = f \delta_0(n) \text{ in } \Omega^\pm$$

Near field:

$$(2) \quad u^\delta = \sum_{n \in \mathbb{N}} \delta^n U_n(\mathcal{V}, S; \theta)$$

$$S = \frac{R_0 \theta}{\delta} \quad \mathcal{V} = \frac{r - R_0}{\delta}, \quad U_n \text{ 1-periodic in } S$$



$$\frac{\partial}{\partial S} \left( \mu \frac{\partial U_n}{\partial S} \right) + \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial U_n}{\partial \mathcal{V}} \right) = \sum_{j=1}^4 \mathcal{A}_j U_{n-j} \text{ in } B_0$$

where  $\mathcal{A}_j$  are differential operators in  $S, \mathcal{V}, \theta$ .

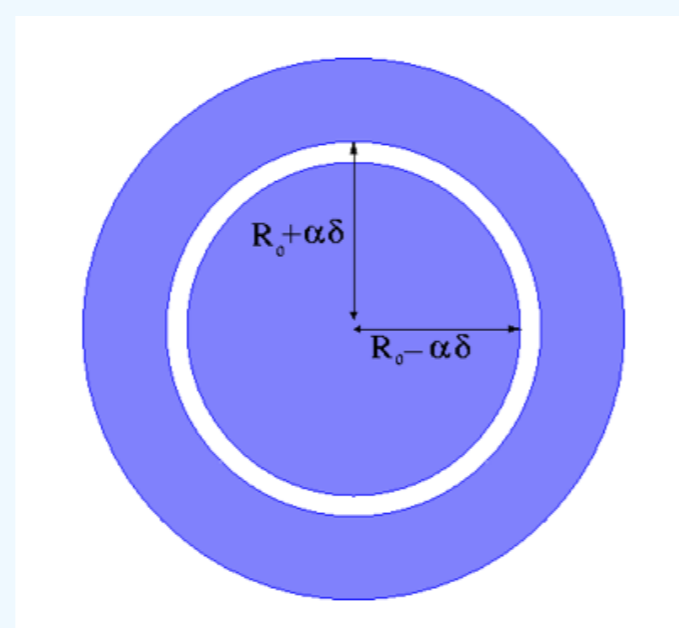
+ matching conditions: (1) and (2) coincide in an overlapping area.

### Approximate Model

- **Main idea**: we want to find an approximate well-posed problem whose solution  $u_1^\delta$  is closed to the two first terms of the far field asymptotic expansion  $u_0 + \delta u_1$ .

- **Method**: we use the fact that  $[u_1^\delta] \approx [u_0] + \delta [u_1]$ .

$$\mathcal{P} \begin{cases} \Delta u_1^\delta + \frac{\omega^2 \rho_\infty}{\mu_\infty} u_1^\delta = f \text{ in } \Omega_{\alpha\delta}^\pm \\ [u_1^\delta]_\alpha = \delta A_0^\alpha \langle r \frac{\partial u_1^\delta}{\partial r} \rangle_\alpha \\ \left[ r \frac{\partial u_1^\delta}{\partial r} \right]_\alpha = \delta \left( B_0^\alpha \langle u_1^\delta \rangle_\alpha + B_2^\alpha \left\langle \frac{\partial^2 u_1^\delta}{\partial \theta^2} \right\rangle_\alpha \right) \\ \partial_r u_1^\delta + i\omega u_1^\delta = 0 \text{ on } S_{R_c} \end{cases}$$



$$\Omega_{\alpha\delta} = \Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-$$

where  $g_\alpha^\pm = g(R_0 \pm \alpha\delta)$ ,  $[g]_\alpha = g_\alpha^+ - g_\alpha^-$ ,  $\langle g \rangle_\alpha = \frac{1}{2}(g_\alpha^+ + g_\alpha^-)$  and

$$A_0^\alpha = \frac{2\alpha - 1}{R_0} + \int_{-1/2}^{1/2} \frac{W_0^0(S, \frac{1}{2}) - W_0^0(S, -\frac{1}{2})}{R_0} dS \quad B_2^\alpha = \frac{1 - 2\alpha}{R_0} - \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{\mu}{\mu_\infty} \left( \frac{\partial V_1^1}{\partial S} + \frac{1}{R_0} \right) dS d\mathcal{V}$$

$$B_0^\alpha = \frac{(1 - 2\alpha)\omega^2 R_0 \rho_\infty}{\mu_\infty} - \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{\omega^2 R_0 \rho}{\mu_\infty} dS d\mathcal{V}$$

where  $V_1^1$  and  $W_0^0$  are solutions of Laplace canonical problems posed in  $B_0$ :

$$\begin{cases} \frac{\partial}{\partial S} \left( \mu \frac{\partial V_1^1}{\partial S} \right) + \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial V_1^1}{\partial \mathcal{V}} \right) = -\frac{1}{R_0} \frac{\partial \mu}{\partial S} \\ V_1^1(R, Z) \sim C \text{ when } \mathcal{V} \rightarrow \pm\infty \end{cases} \quad \begin{cases} \frac{\partial}{\partial S} \left( \mu \frac{\partial W_0^0}{\partial S} \right) + \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial W_0^0}{\partial \mathcal{V}} \right) = 0 \\ \partial_\nu W_0^0 \sim 1 \text{ when } \mathcal{V} \rightarrow \pm\infty \end{cases}$$

**Proposition**: Problem  $\mathcal{P}$  is well-posed as soon as  $B_1^\alpha < 0$  and  $A_0^\alpha > 0$ , i.e when  $\alpha$  exceeds a critical value  $\alpha^* > 0$ . Moreover, for any  $\alpha > \alpha^*$ , for any  $\gamma > 0$ , there exists  $C(\alpha) > 0$ , such that, for  $\delta \leq \delta_0$

$$\|u^\delta - u_1^\delta\|_{H^1(\Omega_\gamma)} \leq C(\alpha) \delta^2$$

### Numerical Results

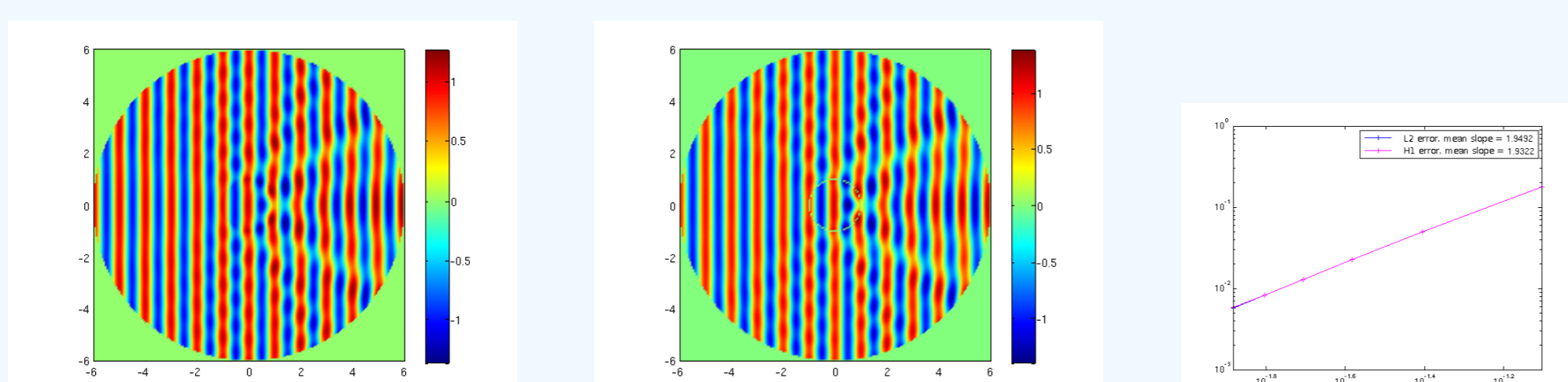
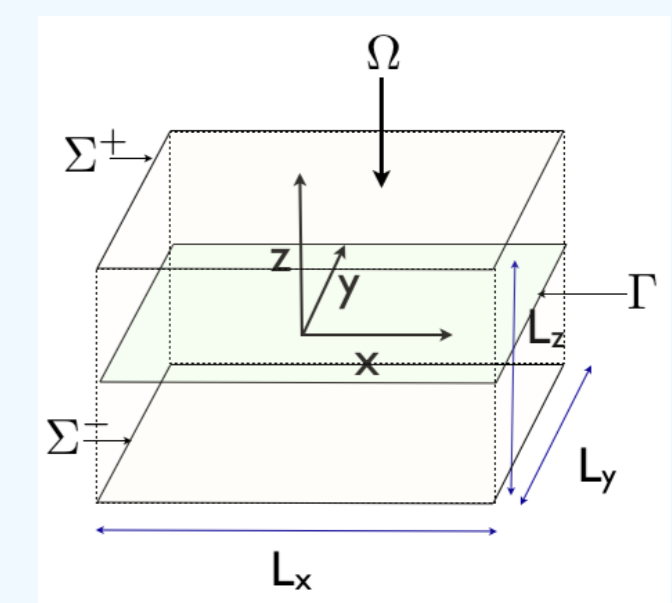


Figure 1: Scattering of a plane wave : 'exact' solution, approximate solution, convergence rate

## 3D Maxwell Problem

### Description

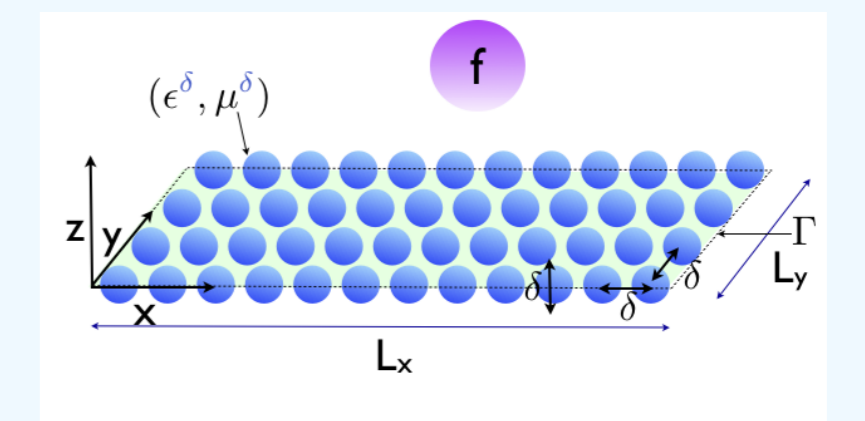
$$\begin{cases} \text{curl} \left( \frac{1}{\mu^\delta} \text{curl} \mathbf{E}^\delta \right) - \omega^2 \epsilon^\delta \mathbf{E}^\delta = F \text{ in } \Omega \\ \mathbf{E}^\delta \text{ } L_x\text{-periodic in } x \\ \mathbf{E}^\delta \text{ } L_y\text{-periodic in } y \\ \frac{1}{i\omega\mu_\infty} \text{curl} \mathbf{E}^\delta \times \mathbf{n} - \mathbf{E}_T^\delta = 0 \text{ on } \Sigma^\pm \end{cases}$$



$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3, |x| \leq \frac{L_x}{2}, |y| \leq \frac{L_y}{2}, |z| \leq \frac{L_z}{2} \right\}$$

-  $(\epsilon^\delta, \mu^\delta)$   $\delta$ -periodic in  $x$  and  $y$

-  $(\mu^\delta, \epsilon^\delta) = (\mu_\infty, \rho_\infty)$  if  $|z| \geq 1/2$

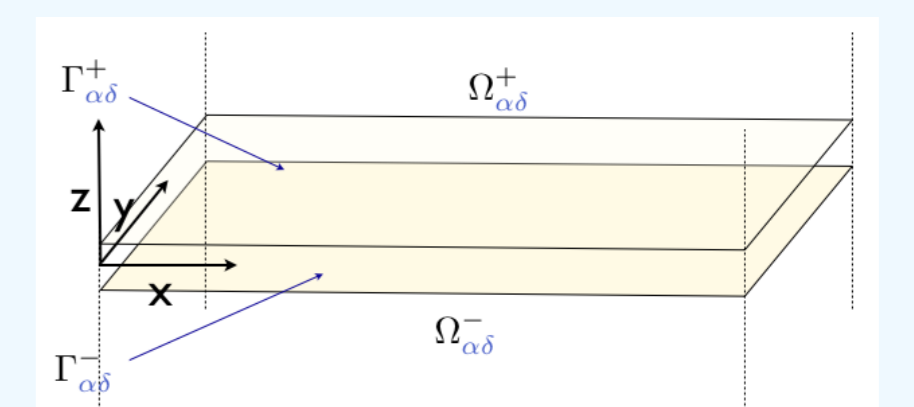


### Approximate Model

$$\mathcal{P}_1 \begin{cases} \text{curl} \text{curl} \mathbf{E}_1^\delta - \omega^2 \mathbf{E}_1^\delta = F \text{ in } \Omega_{\alpha\delta}^\pm \\ [n \times \mathbf{E}_1^\delta]_\alpha + \delta \left( \frac{a^\alpha}{\omega^2} \text{curl}_\Gamma \text{curl}_\Gamma \lambda - D_1^\alpha \lambda \right) = 0 \\ [n \times \text{curl} \mathbf{E}_1^\delta]_\alpha + \delta \left( b^\alpha \text{curl}_\Gamma \text{curl}_\Gamma \langle (\mathbf{E}_1^\delta)_T \rangle_\alpha - \omega^2 D_2^\alpha \langle (\mathbf{E}_1^\delta)_T \rangle_\alpha \right) \\ n \times \text{curl} \mathbf{E}_1^\delta = -i\omega\mu_\infty \mathbf{E}_{1T}^\delta \text{ on } \Sigma^\pm \\ \lambda = \langle \text{curl}(\mathbf{E}_1^\delta)_T \rangle \end{cases}$$

For  $\alpha$  large enough,

- $D_1^\alpha$  and  $D_2^\alpha$  are positive diagonal matrices.
- $a^\alpha, b^\alpha$  are positive constants.



### Existence, Uniqueness, and Stability

$$\text{Operator } G: \begin{cases} H^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow H_t(\text{curl}_\Gamma, \Gamma) \\ f \mapsto \lambda \text{ such that } a^\alpha \text{curl}_\Gamma \text{curl}_\Gamma \lambda - \omega^2 D_1^\alpha \lambda = f \end{cases}$$

→ Assumption :  $\omega$  is such that  $G$  is well defined.

**Variational formulation**:  $X := \left\{ \mathbf{E} \in H_{\text{per}}(\text{curl}, \Omega_{\alpha\delta}^\pm), \langle \mathbf{E}_T \rangle_\alpha \in H(\text{curl}_\Gamma, \Gamma), \mathbf{E}_t \in L_t^2(\Sigma^\pm) \right\}$

$$(\mathcal{P}_1) \Leftrightarrow a^+ \langle \mathbf{E}_1^\delta, \varphi \rangle + b \langle \mathbf{E}_1^\delta, \varphi \rangle = \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} F \cdot \bar{\varphi} dx \quad \forall \varphi \in X$$

$$a^+(E, \varphi) := \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} \text{curl} E \cdot \overline{\text{curl} \varphi} dx + \delta b^\alpha \int_\Gamma \text{curl}_\Gamma \langle E_T \rangle_\alpha \cdot \overline{\langle \varphi \rangle_\alpha} ds - i\omega \int_{\Sigma^+ \cup \Sigma^-} E_T \cdot \overline{\varphi_T} ds$$

$$b(E, \varphi) := -\omega^2 \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} E \cdot \bar{\varphi} dx - \delta \omega^2 \int_\Gamma D_2^\alpha \langle E_T \rangle_\alpha \cdot \overline{\langle \varphi \rangle_\alpha} ds + \frac{\omega^2}{\delta} \langle G([n \times E]), [n \times \varphi] \rangle$$

**Proposition**:  $\mathcal{P}_1$  is well posed. Moreover, there exist  $C > 0$  and  $\delta_0 > 0$  such that, for any  $\delta \leq \delta_0$ ,

$$\| \mathbf{E}_1^\delta \|_{H(\text{curl}, \Omega_{\alpha\delta}^+)} + \| \mathbf{E}_1^\delta \|_{H(\text{curl}, \Omega_{\alpha\delta}^-)} \leq C \left( \|F\|_{L^2(\Omega_{\alpha\delta}^+)} + \|F\|_{L^2(\Omega_{\alpha\delta}^-)} \right)$$

**Main ideas of the proof** (based on the ideas of [4])

- Helmholtz decomposition:  $X = X_0 \oplus \nabla S$

$$S := \left\{ p \in H_{\text{per}}^1(\Omega_{\alpha\delta}^+) \cap H_{\text{per}}^1(\Omega_{\alpha\delta}^-), \langle p \rangle_\alpha \in H_{\text{per}}^1(\Gamma), p|_{\Sigma^\pm} = 0 \right\}$$

$$X_0 := \left\{ E \in X, \text{div} E = 0 \in \Omega_{\alpha\delta}^\pm, \omega^2 \langle E \cdot n \rangle_\alpha = -\frac{\omega^2}{\delta} \text{div}_\Gamma (n \times G([n \times E]_\alpha)), \right.$$

$$\left. -[n \cdot E]_\alpha = \delta \text{div}_\Gamma D_2^\alpha \langle E_T \rangle_\alpha \right\}$$

- Uniform estimate: proof by contradiction  $\Rightarrow$  uniqueness.

- Existence: compactness of  $b$  on  $X_0$ , Fredholm Alternative.

### References

- [1] M. Van Dyke. *perturbation methods in fluid mechanics*. Academic press, 1964.
- [2] G. Allaire. *Shape Optimization by the Homogenization Method*. Springer.
- [3] Yves Achdou. Etude de la réflexion d'une onde électromagnétique par un métal recouvert d'un revêtement métallisé. Technical report, INRIA, 1989.
- [4] Peter Monk. *Finite Element Methods for Maxwell's Equations*. Oxford science publications, 2003.