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Radial orbit instability as a dissipation-induced phenomenon

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ABSTRACT
This paper is devoted to Radial Orbit Instability in the context of self-gravitating dynamical systems. We present this instability in the new frame of Dissipation-Induced Instability theory. This allows us to obtain a rather simple proof based on energetics arguments and to clarify the associated physical mechanism.

Key words: gravitation – stellar dynamics – methods: analytical – instabilities

1 INTRODUCTION

Instabilities in self-gravitating systems are fundamental processes to understand the shape and physical properties of objects such as galaxies or globular clusters. So far, only a few of such mechanisms are described in literature, namely Jeans instability, which governs the collapse of homogeneous systems; gravothermal catastrophe, which concerns isothermal spheres; and radial orbit instability, which occurs in anisotropic, strongly radial spherical systems. If the first two are well understood, and have taken their place in the study of dynamical stellar systems (see Binney & Tremaine 1987, sections 5.2 and 7.3), as a fact, the situation of radial orbit instability is less clear. A complete story of this physical process, spanning almost forty years, is presented in Maréchal & Perez (2009). Three main points stick out (see the review for all detailed references): there is as yet no simple analytical proof of this phenomenon; there is no global consensus about its actual physical mechanism; and yet it is a fundamental process which affects the phase space distribution of primordial galaxies and contributes to produce the radial density profile of evolved systems. The present paper will address the first two of these points.

1.1 Collisionless Boltzmann–Poisson System

We consider a system constituted of a large number N of gravitating particles interacting together. We will assume that all those particles have the same mass m. We denote as $\mathbf{q}$ and $\mathbf{p}$ the position and the associated impulsion of a particle with respect to some Galilean frame $\mathcal{R}$, and $\mathbf{\Gamma} = (\mathbf{q}, \mathbf{p})$ the corresponding point in the phase space $\mathbb{R}^6$.

We assume that the statistical state of the system is described at each instant t by a distribution function $f(\mathbf{\Gamma}, t)$, with $f(\mathbf{\Gamma}, t)\,\mathrm{d}\mathbf{\Gamma}$ representing the number of particles contained in the elementary phase space volume $\mathrm{d}\mathbf{\Gamma}$ located around $\mathbf{\Gamma}$.

If the influence of collisions on the overall dynamics is neglected[1], this distribution function solves the Collisionless Boltzmann–Poisson system (hereafter CBP)

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{q}} f - m\nabla_{\mathbf{q}}\psi\cdot\nabla_{\mathbf{p}} f = \frac{\partial f}{\partial t} + \{f, E\} = 0 \\ \nabla^2_{\mathbf{q}}\psi = 4\pi G m \displaystyle\int f \mathrm{d}\mathbf{p} \end{array} \right.$$

with boundary conditions $\psi =_{|\mathbf{q}|\to+\infty} O\left(r^{-1}\right)$ and $\lim_{|\mathbf{q}|,|\mathbf{p}|\to+\infty} f = 0$. The function $\psi(\mathbf{q}, t)$ is the gravitational potential created by the particles,

$$E(\mathbf{q}, \mathbf{p}, t) := \frac{\mathbf{p}^2}{2m} + m\psi$$

is the one-particle Hamiltonian, and $\{.,.\}$ denotes the Poisson Bracket defined by

$$\{f_1, f_2\} = \nabla_{\mathbf{q}} f_1\cdot\nabla_{\mathbf{p}} f_2 - \nabla_{\mathbf{q}} f_2\cdot\nabla_{\mathbf{p}} f_1$$

Any stationary solution $f_0(\mathbf{\Gamma})$ of the CBP system is associated to an equilibrium state of the particles distribution. It is now well known that CBP system is Hamiltonian with respect to a Poisson bracket of non-canonical form arising from the fact that a distribution function does not constitute a set of canonical field variables (see Kandrup 1990; Perez & Aly 1996): the set of distribution functions is an infinite-dimensional space. The total energy associated to a distribution function f can be written as

$$H[f] = \int \mathrm{d}\mathbf{\Gamma}\frac{\mathbf{p}^2}{2m} f(\mathbf{\Gamma}, t) - \frac{1}{2}\int \mathrm{d}\mathbf{\Gamma}\int \mathrm{d}\mathbf{\Gamma}'\frac{f(\mathbf{\Gamma}, t)\, f(\mathbf{\Gamma}', t)}{|\mathbf{q}-\mathbf{q}'|}$$

For any two functionals $A[f]$ and $B[f]$ of the distribution function, let $\langle A, B\rangle$ denote the Morrison bracket – introduced in the context of plasma physics by Morrison

[1] For a self-gravitating system with large values of N, this hypothesis is justified: see Binney & Tremaine (1987), part 1.2.1.

(1980) – defined by

$$\langle A,B\rangle = \int \mathrm{d}\mathbf{\Gamma} \left\{ \frac{\delta A}{\delta f}, \frac{\delta B}{\delta f} \right\} f$$

where $\frac{\delta A}{\delta f}$ stands for the functional derivative of A, which is the linear part of $A[f+\delta f] - A[f]$. One can easily obtain the Hamiltonian formulation of CBP system

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \langle F,H\rangle$$

where $F[f]$ is any functional of f.

1.2 The stability problem

The stability of equilibrium states is a very old problem of theoretical stellar dynamics, and a large variety of methods has been used to tackle it. A clear consensus was found about the global stability of isotropic spherical systems with distribution function $f_0(E)$ monotonically decreasing: after the pioneering works by Antonov (1961), linear stability was obtained using energy methods after a long series of papers by Kandrup & Sygnet (1985) (and references within) or see also Perez & Aly (1996) for a comparison of the different results; using direct normal mode techniques, complicated proofs were also obtained (Fridmann & Polyachenko 1984; Palmer 1994). Non-linear stability of such spherical isotropic systems was also proven for some specific models (see Rein 2002, and references within).

The stability of anisotropic spherical systems is a more difficult problem. The distribution function depends both on the one-particle energy E and on the squared one particle angular momentum $L^2 := \mathbf{p}^2\mathbf{q}^2 - (\mathbf{p}\cdot\mathbf{q})^2$. The most general result in this context was obtained by Perez & Aly (1996) and concerns linear stability for the restricted case of preserving perturbations, which includes radially symmetric ones. Non-linear stability is assured for some classes of generalized polytropes for which $f_0(E,L^2) = E^k L^{2p}$ with adapted values of k and p (see Rein (2002) and references within).

Some very technical approaches using normal modes claim linear instability for anisotropic systems composed only of radial orbits (see Fridmann & Polyachenko 1984; Palmer 1994): this is known as the radial orbit instability.

See Merritt & Aguilar (1985) for one of the first relevant numerical approaches, Perez et al. (1996) for an intermediate position or Barnes et al. (2009) and references within for the most recent situation of this problem.

A complete historical account **of radial orbit instability** is given in Maréchal & Perez (2009). **In the next section, we present some key features of this process.**

1.3 Basics of radial orbit instability

Radial orbit instability (hereafter ROI) appears in self-gravitating system dynamics with the pionnering works of Antonov (1973) and Hénon (1973). A decade later Polyachenko & Shukhman (1981) propose a stability criterion based on the ratio of radial over tangential kinetic energies, when it is to small the system must leave its spherical symmetry and form a bar. This work is criticized by Palmer & Papaloizou (1987) which suggest, using normal modes techniques, that ROI can occur for arbitrary small values of the Russian ratio, provided the distribution function of the system is unbounded for orbits with zero angular momentum. This paper is also the first one to propose a relevant physical mechanism to understand ROI based on resonant trapped orbits; we note that this mechanism needs a coupling between orbits.

Several factors show that a radial system needs a "seed" from which ROI can appear. This is developed in detail in Roy & Perez (2004). The general idea is that there has to be a near-equilibrium state, so that coupling between orbits has the time to develop, and the instability to grow. In Mac Millan et al. (1999), density profiles in a power law are considered, which means that the core has the time to stabilize before outer zones collapse, causing ROI to appear; it also shows that adding clumps tends to accelerate the process. A counterexample can be found in Trenti & Bertin (2006), which shows that homogeneous spherical haloes do not undergo ROI, as the system tends to isotropy before reaching equilibrium. For this reason, our study will focus on ROI emerging from equilibrium states.

Although ROI is a natural candidate to produce triaxiality which can occur in self-gravitating systems, it was noted (*e.g.* Katz 1991) that this spatial counterpart of ROI could disapear during the merging process of the galaxy formation. However, Huss et al. (1999) and more recently Mac Millan et al. (1999) have shown that ROI is a fundamental initial process which shapes the phase space of the galaxy progenitor and allows it to get the good final mass density profile. In this context, it is therefore important to understand fully the nature of ROI.

In this context, the objective of this paper is twofold. On the first hand, in section 2, we present a general method for investigate instability of self-gravitating systems. This approach couples a general mathematical result by Bloch et al. (1994) which generalizes Lyapunov theory, and the symplectic approach of the stability problem of CBP system (see Bartolomew 1971; Kandrup 1990; and Perez & Aly 1996) – it must be noted that Kandrup (1991) has already used this technique for non spherical systems without the complete mathematical background.

On the second hand, in section 3 we apply this method to obtain a direct energy proof of the radial orbit instability when the system can dissipate energy.

2 DISSIPATION-INDUCED INSTABILITIES AND SELF GRAVITATING SYSTEMS

2.1 The method of energy variation

Consider the first-order variation of an equilibrium $f_0 \to f_0 + f^{(1)}$. It is well-known (see Bartolomew (1971), Kandrup (1990) and Perez & Aly (1996)) that there exists a phase space function $g(\mathbf{\Gamma},t)$, such that

$$f^{(1)}(\mathbf{\Gamma},t) = -\{g,f_0\} \tag{1}$$

This function is called a generator[2] of the perturbation. Written in this form, $f^{(1)}$ is the largest class of physical perturbations which can be considered as acting on f_0. In other words, $f^{(1)}$ is a deformation of f_0 and then there exists a g such that we have equation (1). Associated to this perturbation, variation of the total energy – which turns out to be of second order in g, see for instance a short calculation in Maréchal & Perez (2009) – is given by

$$H^{(2)}[f_0] = -\int \{g,E\}\{g,f_0\}\mathrm{d}\mathbf{\Gamma} - Gm^2 \iint \frac{\{g,f_0\}\{g',f_0'\}}{|\mathbf{q}-\mathbf{q}'|}\mathrm{d}\mathbf{\Gamma}\,\mathrm{d}\mathbf{\Gamma}' \qquad (2)$$

where $\mathbf{\Gamma}'$ refers to $(\mathbf{q}',\mathbf{p}')$, f_0' to $f_0(\mathbf{\Gamma}')$ and so on.

When $H^{(2)}[f_0]$ is positive for a given set $\mathbb{G}$ of acceptable generators g, the system is reputed stable against the associated perturbations. This argument was detailed and used to prove, in the case when $f_0 = f_0(E)$ and $\partial_E f_0 := \frac{\partial f_0}{\partial E} < 0$, stability against all acceptable g; and when $f_0 = f_0(E, L^2)$ and $\partial_E f_0 < 0$, stability for all g such that $\{g, L^2\} = 0$ which are called preserving perturbations (see Perez & Aly 1996 for all details).

When there are negative energy modes, generators g that cause $H^{(2)}[f_0] < 0$, they are not necessarily associated to an instability. Taking into account dissipation in the system can drastically change its dynamics.

In the next section, we will illustrate this point with a simpler, yet instructive example.

2.2 An electromagnetic example of dissipation-induced instability

Consider a particle of mass $m = 1$, charge e, let us denote as $\mathbf{q} = (x,y,z)^\top$ its position with respect to some Galilean frame. This particle is influenced by two forces: one derives from a potential V that is maximal at $\mathbf{q} = 0$. We'll write $V(\mathbf{q}) = -\frac{1}{2}\omega^2\mathbf{q}^2$. The other one is the Lorentz force generated by a static magnetic field $\mathbf{B} = B_0\mathbf{e}_z = \nabla\wedge\mathbf{A}$ with $\mathbf{A} = \frac{B_0}{2}(x\mathbf{e}_y - y\mathbf{e}_x)$. The Lagrangian of this particle is $\mathcal{L} = \frac{1}{2}\dot{\mathbf{q}}^2 + e\dot{\mathbf{q}}\cdot\mathbf{A} + \frac{1}{2}\omega^2\mathbf{q}^2$, then the impulsion $\mathbf{p}$ conjugate to the position $\mathbf{q}$ is given by $\mathbf{p} = \nabla_{\dot{\mathbf{q}}}(\mathcal{L}) = (p_x,p_y,p_z)^\top$, and thus, with $\beta = \frac{eB_0}{2}$:

$$\left\{\begin{array}{l} p_x = \dot{x} - \beta y \\ p_y = \dot{y} + \beta x \\ p_z = \dot{z} \end{array}\right.$$

The Hamiltonian of the system is $\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{q}} - \mathcal{L}$, the equations of motion are given by Hamilton's ones, i.e. $\dot{\mathbf{q}} = \nabla_{\mathbf{p}}(\mathcal{H})$ and $\dot{\mathbf{p}} = -\nabla_{\mathbf{q}}(\mathcal{H})$. The behaviour of (z,p_z) being trivial and independent from movement on the other axes, let us focus on the system in the reduced phase space of $\xi = (x,y,p_x,p_y)^\top$ for which one has

$$\dot{\xi} = \Lambda\xi \quad \text{where} \quad \Lambda = \left(\begin{array}{cc} \beta K & I_2 \\ \alpha I_2 & \beta K \end{array}\right) \qquad (3)$$

$$\text{with} \quad K = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \quad \alpha = \omega^2 - \beta^2 \qquad (4)$$

Since $K^2 = -I_2$ one can see that if we split $\Lambda = A + B$ with

$$A = \left(\begin{array}{cc} 0 & I_2 \\ \alpha I_2 & 0 \end{array}\right) \quad \text{and} \quad B = \beta\left(\begin{array}{cc} K & 0 \\ 0 & K \end{array}\right)$$

we have the fundamental property $AB = BA$, so $\exp(\Lambda t) = \exp(At)\exp(Bt)$, by direct series summation one can find that

$$\exp(Bt) = \left(\begin{array}{cc} \Sigma(t) & 0 \\ 0 & \Sigma(t) \end{array}\right)$$

with

$$\Sigma(t) = \left(\begin{array}{cc} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{array}\right) \in SO_2(\mathbb{R})$$

and

$$\exp(At) = \left(\begin{array}{cc} \varphi_1(t) I_2 & \varphi_2(t) I_2 \\ \alpha\varphi_2(t) I_2 & \varphi_1(t) I_2 \end{array}\right)$$

with

$$\left\{\begin{array}{lcl} \varphi_1(t) & = & \cosh(\sqrt{\alpha}t) \\ \varphi_2(t) & = & (\alpha)^{-1/2}\sinh(\sqrt{\alpha}t) \end{array}\right. \quad \textbf{if} \quad \alpha > 0$$

$$\left\{\begin{array}{lcl} \varphi_1(t) & = & \cos(\sqrt{-\alpha}t) \\ \varphi_2(t) & = & (-\alpha)^{-1/2}\sin(\sqrt{-\alpha}t) \end{array}\right. \quad \textbf{if} \quad \alpha \leq 0$$

The general solution of the problem then writes

$$\xi(t) = \exp(At)\cdot\exp(Bt)\cdot\xi(t=0)$$

and it is stable provided that $\alpha = \omega^2 - \beta^2 \leq 0$. We note that this stability is not asymptotic as all eigenvalues of Λ lie on the imaginary axis. The mathematical condition on α corresponds to the physical case when the effect of the magnetic field $\mathbf{B}$ is stronger than the effect of the scalar potential V. We can see on this example that it is possible to have a stable equilibrium even on a point where the potential is at a maximum; *negative energy variations around an equilibrium is not a sufficient criterion for an instability.* The physical explanation is that the magnetic force, which does not derive from a scalar potential, tends to 'curve' the particle's trajectory, and if the magnetic field is strong enough, this can be enough to keep the particle close to the potential maximum in spite of the repulsive force.

However, this behaviour is only possible as long as there is no energy dissipation. If the system is able to dissipate energy, such an equilibrium becomes unstable. For example, assume there is some form of fluid friction force $\mathbf{F}_f = -\gamma\dot{\mathbf{q}}$, the system is no longer Hamiltonian. Movement is still trivial in the (z,p_z) plane; keeping the same variables that we have used in the non-dissipative case, the equation of motion is now

$$\dot{\xi} = \Lambda_\gamma\xi \quad \text{where} \quad \Lambda_\gamma = \Lambda + \gamma C, \quad C = \left(\begin{array}{cc} 0 & 0 \\ -\beta K & -I_2 \end{array}\right).$$

The matrix C happens to **commute with** B, so we have $(A+\gamma C)B = B(A+\gamma C)$: the fundamental matrix of the dissipative system splits into

$$\exp(\Lambda_\gamma t) = \exp([A+\gamma C]t)\cdot\exp(Bt)$$

As $\exp(Bt)$ is a rotation matrix, the stability of the dynamics is governed by $\exp([A+\gamma C]t)$. The characteristic polynomial of $A_\gamma = A + \gamma C$ is

$$\chi(\lambda) = \lambda^4 + 2\gamma\lambda^3 + (\gamma^2 - 2\alpha)\lambda^2 - 2\alpha\gamma\lambda + \gamma^2\beta^2 + \alpha^2$$

roots of which are

$$\lambda_{1,2} = \frac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 + 4\alpha + 4i\beta\gamma}\right]$$

[2] This function is clearly not unique.

and

$$\lambda_{3,4} = \frac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 + 4\alpha - 4i\beta\gamma}\right].$$

Let us focus on the transition from the stable equilibrium we have determined towards the dissipative case. We then have $\alpha = \omega^2 - \beta^2 < 0$ and $0 < \gamma \ll 1$. In this limit case, one can get

$$\lambda_{1,2} = \frac{\gamma}{2}\left(-1 \pm \left(1 - \frac{\omega^2}{\beta^2}\right)^{-1/2}\right) \pm i\left(\beta^2 - \omega^2\right)^{1/2} + o\left(\gamma\right)$$

and

$$\lambda_{3,4} = -\frac{\gamma}{2}\left(-1 \pm \left(1 - \frac{\omega^2}{\beta^2}\right)^{-1/2}\right) \pm i\left(\beta^2 - \omega^2\right)^{1/2} + o\left(\gamma\right)$$

¿From our assumption that $\alpha < 0$, we have $(1-\frac{\omega^2}{\beta^2})^{-1/2} > 1$, therefore there is a pair of roots with positive real parts: the system becomes *spectrally unstable*. When γ is not infinitesimal, this instability persists as one can check by direct spectrum calculation or by more elegant approaches. The physical meaning is clear: if the particle loses energy, the magnetic field cannot 'curve' it back as close to the maximum as it was previously, and it will spiral further and further from the origin.

2.3 Dissipation-induced instabilities

The previous three-dimensional example is a special case of a general theorem which applies for finite dimensional systems: a Hamiltonian dynamical system with a negative energy mode (which could be stable **without further hypothesis**) becomes spectrally and hence linearly and nonlinearly unstable when any kind of dissipation is introduced. This counterintuitive result takes its genesis from the classical works by Thomson (Lord Kelvin) and Tait (1879), **but** it was proven only recently in the case of finite-dimensional systems (Bloch et al. 1994 and Krechetnikov & Marsden 2007), and, as suggested by references in the latter, appears to be very useful in mechanics. More recent works by Krechetnikov & Marsden (2009) suggest that the infinite-dimensional **case** works similarly**, although there is no definitive proof for the time being.** In the context of theoretical astrophysics, it is interesting to note that H. Kandrup used such kind of arguments to investigate gravitational instabilities for triaxial systems (see Kandrup 1991), before any actual, formal result.

As recalled in section 1.1, CBP *is* a Hamiltonian infinite-dimensional system, so we can apply this theory of dissipation-induced instability for stability investigations in this context of gravitational plasmas. In the next section we will show that, when a spherical and anisotropic self-gravitating system becomes more and more radial, we can choose a certain class of g for which $H^{(2)}[f_0] < 0$: this proves the existence of negative energy modes in such systems. Following the dissipation-induced instability theory such kind of gravitating systems will become unstable as soon as any kind of dissipation can appear. As noticed by Kandrup in his visionary paper, in physical self-gravitating systems dissipation could take several forms like a little bit of gas, dynamical friction or at minimum gravitational radiation! In the context of numerical modelizations of self-gravitating systems where radial orbit instability also appears, dissipation is also inevitably introduced by numerical algorithms of time integration or by potential computation.

3 APPLICATION TO RADIAL ORBIT INSTABILITY

3.1 Approaching a radial system

A pure radial orbit system is characterized by particles with $L^2 = 0$, the corresponding distribution function could then be written $f_0^{\mathbf{ro}}\left(E, L^2\right) = \varphi\left(E\right)\delta\left(L^2\right)$ where φ is any positive smooth normalized function, and δ denotes the Dirac distribution. However, this distribution is very irregular in zero which is quite problematic, in addition to being unrealistic (orbits can hardly be perfectly radial). So, instead of actually using the Dirac distribution, we will use functions that approach it.

The choice we made is to use Gaussian functions. More specifically, we will consider an initial distribution function of the form

$$f_0^a\left(E, L^2\right) = \varphi\left(E\right)\delta_a\left(L^2\right), \quad \delta_a(L^2) = \frac{1}{\pi a^2}\exp\left(-\frac{L^2}{a^2}\right) \tag{5}$$

By direct calculation one can easily check that, for any smooth function Z defined on the phase space, one has, by limited development of Z with respect to p_θ and p_ϕ[3, 4]

$$\begin{aligned}\int Z\delta_a(L^2)\mathrm{d}\Gamma &= \int Z\delta_a(L^2)\frac{\mathrm{d}p_r\,\mathrm{d}p_\theta\,\mathrm{d}p_\phi}{r^2\sin\left(\theta\right)}\,\mathrm{d}^3\mathbf{q}\\ =& \int\frac{1}{r^2}\left(Z + \frac{a^2}{4}\frac{\partial^2 Z}{\partial p_\theta^2} + \frac{a^2}{4}\sin^2(\theta)\frac{\partial^2 Z}{\partial p_\phi^2}\right)\Bigg|_{L^2=0}\mathrm{d}p_r\,\mathrm{d}^3\mathbf{q}\\ &+O(a^4)\end{aligned} \tag{6}$$

which has a clear limit when $a \to 0$ and selects the value of Z at $L^2 = 0$ as expected. We could say that f_0^a tends to a distribution function of purely radial orbits when $a \to 0$.

In the following part, we will consider what happens for arbitrarily small values of a.

3.2 Energy variation

Our goal in this section is to show that there exist perturbation generators g that, for sufficiently small values of a

[3] N.B.: variables p_r, p_θ and p_ϕ thereafter are the conjugate variables of r, θ and ϕ, not the projections of $\mathbf{p}$ along the base vectors. We have $p_r = m\dot{r}$, $p_\theta = mr^2\dot{\theta}$ and $p_\phi = mr^2\sin^2(\theta)\dot{\phi}$. Also

$$L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2(\theta)}$$

[4] The calculation involves the well-known Gaussian integrals

$$\begin{aligned}\int e^{-\frac{x^2}{r^2}}\mathrm{d}x &= r\sqrt{\pi}\\ \int x^2 e^{-\frac{x^2}{r^2}}\mathrm{d}x &= \frac{1}{2}r^3\sqrt{\pi}\\ \int x^4 e^{-\frac{x^2}{r^2}}\mathrm{d}x &= \frac{3}{4}r^5\sqrt{\pi}\end{aligned}$$

(that is, for systems that are close enough to the purely radial case), give a negative energy variation. To do so, we will start with a general g, calculate the energy variation $H^{(2)}[f_0^a]$ for our quasi-radial systems, and explain along the way what hypotheses we make about g to reach this goal.

Usual Poisson bracket properties give, for $f_0^a(E,L^2)$

$$\{g,f_0^a\}=\partial_E f_0^a\{g,E\}+\partial_{L^2}f_0^a\{g,L^2\}$$

where $\partial_E f_0^a := \frac{\partial f_0^a}{\partial E}$ and $\partial_{L^2} f_0^a := \frac{\partial f_0^a}{\partial L^2}$, hence the second order energy variation (2) splits into

$$H^{(2)}[f_0^a]=K_{L^2}+K_E-G\iint\frac{(\delta\rho_{L^2}+\delta\rho_E)(\delta\rho'_{L^2}+\delta\rho'_E)}{|\mathbf{q}-\mathbf{q}'|}\mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{q}' \quad (7)$$

where

$$K_{L^2} := -\int\partial_{L^2}f_0^a\{g,E\}\{g,L^2\}\mathrm{d}\mathbf{\Gamma} \quad (8)$$

$$K_E := -\int\partial_E f_0^a\{g,E\}^2\mathrm{d}\mathbf{\Gamma} \quad (9)$$

$$\delta\rho_{L^2} := -m\int\partial_{L^2}f_0^a\{g,L^2\}\mathrm{d}^3\mathbf{p} \quad (10)$$

$$\delta\rho_E := -m\int\partial_E f_0^a\{g,E\}\mathrm{d}^3\mathbf{p} \quad (11)$$

For a general perturbation, it is difficult to say more about the sign of $H^{(2)}$. However, the system could receive any kind of perturbations. In order to go further, we have to make some assumptions about g. We already know, from section 1.2, that a radial function will not lead to an instability, and from section 2.1, that dependency on E and L^2 plays no part. To find a g function that works, we thus have to consider a non-radial perturbation. We can consider a perturbation that is axisymmetric around the z axis:

$$g(\mathbf{\Gamma})=g(E,L^2,\theta,p_\theta) \quad (12)$$

With this hypothesis:

$$\{g,L^2\} = 2p_\theta\frac{\partial g}{\partial\theta}+2p_\phi^2\frac{\cos(\theta)}{\sin^3(\theta)}\frac{\partial g}{\partial p_\theta} \quad (13)$$

$$\{g,E\} = \frac{1}{2mr^2}\{g,L^2\} \quad (14)$$

With $f_0^a=\varphi(E)\delta_a\left(L^2\right)$, we get

$$\partial_E f_0^a = \varphi'(E)\delta_a(L^2) \quad (15)$$

$$\partial_{L^2}f_0^a = -\frac{1}{a^2}\varphi(E)\delta_a(L^2) \quad (16)$$

With the previous results, and using (6), we can calculate explicitly the four terms K_{L^2}, K_E, $\delta\rho_{L^2}$ and $\delta\rho_E$ in a power series of a for $a\to 0$. If we consider only the first term in a, which corresponds to the term of lower power in p_θ and p_ϕ, a long but straightforward calculation eventually leads to the following results:

$$K_{L^2} = \frac{1}{m}\int\frac{1}{r^4}\varphi(E)\left(\frac{\partial g}{\partial\theta}\right)^2\Big|_{L^2=0}\mathrm{d}p_r\,\mathrm{d}^3\mathbf{q} \quad (17)$$

$$K_E = -\frac{a^2}{m^2}\int\frac{1}{2r^6}\varphi'(E)\left(\frac{\partial g}{\partial\theta}\right)^2\Big|_{L^2=0}\mathrm{d}p_r\,\mathrm{d}^3\mathbf{q} \quad (18)$$

$$\delta\rho_{L^2} = \frac{m}{r^2}\int\varphi(E)\left(\frac{\partial^2 g}{\partial\theta\partial p_\theta}+\frac{\cos(\theta)}{\sin(\theta)}\frac{\partial g}{\partial p_\theta}\right)\Big|_{L^2=0}\mathrm{d}p_r \quad (19)$$

$$\delta\rho_E = -\frac{ma^2}{2r^4}\int\varphi'(E)\left(\frac{\partial^2 g}{\partial\theta\partial p_\theta}+\frac{\cos(\theta)}{\sin(\theta)}\frac{\partial g}{\partial p_\theta}\right)\Big|_{L^2=0}\mathrm{d}p_r \quad (20)$$

As one can see K_E and $\delta\rho_E$ are of order a^2 for $a\to 0$, whereas K_{L^2} and $\delta\rho_{L^2}$ do not depend on a in the same regime. Therefore, for near radial orbit systems one can neglect K_E in front of K_{L^2} and $\delta\rho_E$ in front of $\delta\rho_{L^2}$.

The second-order energy variation of perturbed near radial orbit systems is then

$$\begin{aligned}H^{(2)}[f_0^a] &= \frac{1}{m}\int\frac{\varphi(E)}{r^4}\left(\frac{\partial g}{\partial\theta}\right)^2\Big|_{L^2=0}\mathrm{d}p_r\,\mathrm{d}^3\mathbf{q}\\ &- G\iint\frac{\delta\rho_{L^2}\delta\rho'_{L^2}}{|\mathbf{q}-\mathbf{q}'|}\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{q}'\end{aligned} \quad (21)$$

The first term is clearly positive as an integral of a positive function, while the second is clearly negative owing to the negativeness of the Laplacian operator: introducing

$$\mu(\mathbf{q}) := -\int\frac{\delta\rho'_{L^2}}{|\mathbf{q}-\mathbf{q}'|}\mathrm{d}^3\mathbf{q}' \quad (22)$$

one has $\Delta\mu=4\pi\delta\rho_{L^2}$, hence

$$\begin{aligned}-\iint\frac{\delta\rho_{L^2}\delta\rho'_{L^2}}{|\mathbf{q}-\mathbf{q}'|}\mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{q}' &= \frac{1}{4\pi}\int\mu\Delta\mu\mathrm{d}\mathbf{q}\\ &= -\frac{1}{4\pi}\int(\nabla\mu)^2\mathrm{d}^3\mathbf{q}<0\end{aligned}$$

The sign of $H^{(2)}$ is thus unclear, unless we make another assumption about g. To be able to say more, we can consider a generating function verifying

$$\forall E,\ \theta\ :\quad \frac{\partial g}{\partial\theta}\Big|_{L^2=0}=0 \quad\text{while}\quad \frac{\partial g}{\partial p_\theta}\Big|_{L^2=0}\neq 0 \quad (23)$$

For such perturbations one can easily check that $K_{L^2}=0$ and $\delta\rho_{L^2}\neq 0$, therefore

$$H^{(2)}[f_0^a]=-G\iint\frac{\delta\rho_{L^2}\delta\rho'_{L^2}}{|\mathbf{q}-\mathbf{q}'|}\mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{q}'<0 \quad (24)$$

To summarize, in this section, we have obtained:

Result 1. Let f_0^a be a distribution function of nearly-radial orbits, such that

$$f_0^a=\varphi(E)\frac{1}{\pi a^2}\exp\left(-\frac{L^2}{a^2}\right).$$

Let g be a generator of a perturbation, of the form $g(\mathbf{\Gamma})=g(E,L^2,\theta,p_\theta)$, with $\frac{\partial g}{\partial\theta}\big|_{L^2=0}=0$ while $\frac{\partial g}{\partial p_\theta}\big|_{L^2=0}\neq 0$.

Then for sufficiently small values of a, the energy variation $H^{(2)}[f_0^a]$ caused by g is negative.

3.3 Density variation

It is interesting to analyse the density variation associated to the generator described in the previous section. The symplectic formulation of the problem allows us to write the perturbation of the distribution function in terms of the generating function g: this is equation (1). ¿From this relation one can obtain the density

$$\begin{aligned}\rho(\mathbf{q}) &= \rho_0+\rho^{(1)}=\int mf\mathrm{d}\mathbf{p}\\ &= \int mf_0^a\mathrm{d}^3\mathbf{p}-\int m\{g,f_0^a\}\mathrm{d}^3\mathbf{p}\\ &= \int mf_0^a\mathrm{d}^3\mathbf{p}+\delta\rho_E+\delta\rho_{L^2}\end{aligned}$$

For sufficiently small values of a, $\delta\rho_E$ is negligible in front of $\delta\rho_{L^2}$, as we have seen in (20), hence $\delta^{(1)}\rho = \delta\rho_{L^2}$. Using (19), it can be checked that the first-order variation of total mass $\delta^{(1)}m$ associated to the perturbation is vanishing.

$$\begin{aligned}\delta^{(1)}m &= \int \delta^{(1)}\rho \mathrm{d}^3\mathbf{q} = m\int \delta\rho_{L^2}\, r^2\sin(\theta)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= m\int \varphi(E)\left(\sin(\theta)\frac{\partial^2 g}{\partial\theta\partial p_\theta}+\cos(\theta)\frac{\partial g}{\partial p_\theta}\right)\Bigg|_{L^2=0}\mathrm{d}p_r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= m\int \varphi(E)\left(\int_0^\pi \frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial g}{\partial p_\theta}\right)\mathrm{d}\theta\right)\mathrm{d}p_r\,\mathrm{d}r\,\mathrm{d}\phi \\ &= 0\end{aligned}$$

Without more hypotheses than (23) on the perturbation generating function g, and for sufficiently small values of a, the first order induced variations of density are

$$\delta^{(1)}\rho = \delta\rho_{L^2} = \frac{m}{r^2}\int \varphi(E)\left(\frac{\partial^2 g}{\partial\theta\partial p_\theta}+\frac{\cos(\theta)}{\sin(\theta)}\frac{\partial g}{\partial p_\theta}\right)\Bigg|_{L^2=0}\mathrm{d}p_r$$

In order to obtain some physical characteristics of the instability, we have to make yet another assumption about g. To find a function that verifies condition (23), given the form of g given in (12), we can suppose for example that g is separated in the θ variable, i.e. one can find two functions A and B such that

$$g(E,L^2,\theta,p_\theta) = B(E,L^2,p_\theta)A(\theta) \qquad (25)$$

Under this assumption, criterion (23) becomes

$$\forall E,\ \theta:\quad A'(\theta)B(E,0,0)=0 \quad \text{while} \quad A(\theta)\frac{\partial B}{\partial p_\theta}(E,0,0)\neq 0 \qquad (26)$$

It is very easy to find a B that verifies this condition. A direct calculation then gives

$$\delta^{(1)}\rho = m\frac{D(\theta)}{r^2}\int \varphi(E)|_{L^2=0}\frac{\partial B}{\partial p_\theta}(E,0,0)\mathrm{d}p_r$$

where

$$D(\theta) = A'(\theta)+\frac{\cos(\theta)}{\sin(\theta)}A(\theta)$$

If $D(\theta)$ is not constant, which corresponds to a wide class of A [5], then $\delta^{(1)}\rho$ does depend on θ and the spherical symmetry of the equilibrium state is broken. [6]

We have reached our goal: we have found a class of perturbations g which leads to a negative energy variation, and which creates a density variation that is *not* spherically symmetric. As per section 2, this means that with the help of dissipation, the system is unstable against this perturbation: hence a favoured direction will appear in the system, which was initially spherical.

Result 2. Consider a self-gravitating system, described by the CBP system and represented by a distribution function $f_0(E,L^2)$, that is spherically symmetric and with nearly radial orbits. Assume this system can dissipate energy.

Then there exists perturbations, generated by a function g, against which the system is unstable, and that cause it to lose its spherical symmetry.

[5] The equation $D(\theta)\neq k$ can be easily solved and gives

$$A(\theta)\neq\frac{\lambda - k\cos(\theta)}{\sin(\theta)}$$

where k and λ are θ-free constants.

[6] The fact that the resulting perturbation depends only on θ, r and E, and thus is axisymmetric around the z axis, is of course a consequence of our choice of the form (12) for the generating function.

4 CONCLUSION

In this paper we have shown two important points: self-gravitating dynamical systems described by the Collisionless Boltzmann–Poisson equations are candidates for Dissipation-Induced Instability when they are more and more radially anisotropic; and this mechanism generically introduces a favoured direction in the spatial part of the system's phase space. In comparison with previous tedious normal modes techniques used in this context, the detail of the first point gives a simple proof of radial orbit instability based on energetics arguments. Dissipation, which is needed in our proof, is also implicitly required in the classical intuitive understanding of this instability **presented in Palmer (1994) (section 7.3.1). It is a fact that in a** *pure* **radial system - which is the most unstable - orbits, which are frozen in a fixed direction, cannot precess or librate as it is required for the trapping resonance invoked by Palmer. Hence, if two radial orbits actually attain a lower energy state by approaching each other, this mechanism actually needs a way to dissipate excessive energy.** Finally, a point about time-scales should be stressed: it is well known that radial orbit instability is effective on a few crossing times, therefore if dissipation appears to be the cornerstone of radial orbit instability, it is clear that it could not act alone. Non-linear and non-local aspects of the gravitational potential clearly amplifies and completes the dissipation-triggered work.

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