Acoustic propagation in non uniform waveguides: revisiting Webster equation using evanescent boundary modes

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The scattering of an acoustic wave propagating in a non uniform waveguide is inspected by revisiting improved multimodal methods in which the introduction of additional modes, so called boundary modes, allow to better satisfy the Neumann boundary conditions at the varying walls. In this paper, we show that the additional modes can be identified as evanescent modes. Although non physical, these modes are able to tackle the evanescent part of the field omitted by the truncation and are able to restore the right boundary condition at the walls. In the low frequency regime, the system can be solved analytically and the solution for an incident plane wave including one or two boundary modes is shown to be an improvement of the usual Webster equation.

1. Introduction

Since the pioneering work of Stevenson (1951), the multimodal propagation method has been widely used to describe propagation in non uniform waveguides in acoustics (Ravazy 1994, Pagneux *et al.* 1996, Felix & Pagneux 2001, Bi *et al.* 2007, Potel & Bruneau 2008) and in elasticity (Pagneux & Maurel 2002, 2004). In the 2D acoustic case, the pressure p satisfies the Helmholtz equation:

$$(\Delta + k^2)p(x,y) = 0, \qquad (1.1)$$

in a waveguide of height $h(x)$ (located in $y \in [0, h(x)]$) with boundary conditions $\partial_y p(x,0) = 0$ and

$$\partial_y p(x,h) = h'(x)\partial_x p(x,h). \qquad (1.2)$$

In a uniform waveguide of constant height h, the solution $p(x,y)$ is expanded in a series $p(x,y) = \sum_{n=0}^{N} p_n(x)\psi_n(y)$, using the transverse modes of the waveguide $\psi_n(y) = A_n \cos(n\pi y/h)$ (where A_n are normalization coefficients). The function ψ_n satisfies the boundary conditions $\psi_n'(0) = 0$ and $\psi_n'(h) = 0$ (as p). For a non uniform waveguide, $p(x,y)$ is still expanded as a series using the local transverse modes $\psi_n(y;x) = A_n \cos(n\pi y/h(x))$ (ψ_n depends now on x, and we still have $\partial_y \psi_{n|y=h(x)} = 0$). Although modal approaches have been shown to be efficient when sufficient evanescent modes are taken into account (Pagneux *et al.* 1996), this representation of $p(x,y)$ is incompatible with the boundary condition (1.2)

Proc. R. Soc. A 1–27; doi: 10.1098/rspa.00000000

if $h'(x) \neq 0$. Further, for each x value the convergence of the series is poor, the derivative of the series does not converge uniformly in the interval $[0, h(x)]$. In a series of papers, Athanassoulis & Belibassakis (1999, 2003) propose an improved representation in which the function $q(x, y)$ is introduced:

$$q(x, y) \equiv p(x, y) - \partial_y p(x, h)\xi(y; x), \tag{1.3}$$

where ξ is a function satisfying $\partial_y \xi_{|y=h(x)} = 1$ and $\partial_y \xi_{|y=0} = 0$. This new function satisfies $\partial_y q(x, 0) = 0 = \partial_y q(x, h)$ (as ψ_n), so that the series $q(x, y) = \sum_{n=0}^{N} q_n(x)\psi_n(y; x)$, and its derivative with respect to y, converge uniformly in the interval $y \in [0, h(x)]$. However, the value of $\partial_y p(x, h)$ is in general unknown *a priori* so that it is not possible to write p as a function of q.

An exception occurs in the case of an impedance boundary condition of the form $\partial_y p(x, h) = Y_0 p(x, h)$ as considered in Bi *et al.* 2007. Choosing $\xi(h; x) = 0$, we deduce from (1.3) that $q(x, h) = p(x, h)$, leading to $p(x, y) = q(x, y) + Y_0 q(x, h)\xi(y; x)$ and thus the solution can be written:

$$p(x, y) = \sum_{n=0}^{N} q_n(x) \left[\psi_n(y; x) + Y_0 \xi(y; x)\psi_n(h; x)\right].$$

In this case, the convergence of the truncated series $p(x, y)$ is the same as the convergence of the series $q = \sum_{n=0}^{N} q_n\psi_n$ and the boundary condition $\partial_y p(x, h) - Y_0 p(x, h) = 0$ is satisfied strictly for any truncation number N.

When considering the boundary condition (1.2), $\partial_y p(x, h)$ is unknown and the approach must be modified. Athanassoulis & Belibassakis (1999, 2003) propose to keep the additional transverse mode ξ and to introduce an additional unknown modal component $q_{N+1}(x)$ in the expansion:

$$p(x, y) = \sum_{n=0}^{N} q_n(x)\psi_n(y; x) + q_{N+1}(x)\xi(y; x), \tag{1.4}$$

where $q_{N+1}(x)$ is expected *in fine* to behave as $\partial_y p(x, h)$. However, this cannot be guaranteed for any finite truncation. Using this decomposition in (1.1) and (1.2), a system of coupled ordinary differential equations is obtained:

$$\sum_{m=0}^{N+1} A_{nm}(x)q''_m(x) + B_{nm}(x)q'_m(x) + C_{nm}(x)q_m(x) = 0, \tag{1.5}$$

for $n = 0, \ldots N + 1$. The system is found to remain coupled in the straight part of the waveguide which implies that the classical radiation condition $q'_n = \pm i k_n q_n$ cannot be applied directly at the inlet/outlet of the scattering region (where $h' \neq 0$), with the wavenumbers $k_{n\leq N} \equiv \sqrt{k^2 - (n\pi/h)^2}$. In Athanassoulis & Belibassakis (1999, 2003) and Hazard & Lunéville (2008), the system (1.5) has been solved using finite difference methods by imposing $q_{N+1} = 0$ at the inlet/outlet and the usual radiation conditions on the $q_{n\leq N}$. The convergence of the improved representation has been shown (Hazard & Lunéville 2008).

In the present paper, we revisit the coupled mode equations (1.5) in order to derive an improved system adapted to define radiation conditions. To do so we need to define the modal components:

$$p_n \equiv (p, \psi_n),$$

in a orthogonal basis of transverse functions (ψ_n). This is possible, defining from (1.4), the supplementary transverse function $\psi_{N+1} \equiv \xi - \sum_{n=0}^{N} \alpha_n \psi_n$ with $\alpha_n \equiv (\xi, \psi_n)$, $n \leq N$. We get

$$p(x,y) = \sum_{n=0}^{N+1} p_n(x)\psi_n(y;x), \tag{1.6}$$

with

$$\begin{cases} p_{n\leq N} \equiv q_n + \alpha_n q_{N+1}, \\ p_{N+1} \equiv q_{N+1}. \end{cases} \tag{1.7}$$

The resulting coupled mode equations is a partially decoupled system for the (p_n) components, $n=0$ to $N+1$, which is the key point of our new formulation:

$$p_n''(x) + k_n^2 p_n(x) = \sum_{m=0}^{N+1} D_{nm}(x)p_m(x) + E_{nm}(x)p_m'(x), \tag{1.8}$$

where D, E vanish in the straight parts of the waveguide. Now, approximate solutions can be found using successive Born approximations, as in Maurel & Mercier 2012. The leading order solution is found for an incident wave $p^{inc}(x,y)$ and

$$p_n^{(0)} = (p^{inc}, \psi_n)$$

is solution of our system (1.8) for $h'=0$, namely $p_n^{(0)''} + k_n^2 p_n^{(0)} = 0$. Then, this solution can be used as the first step of an iterative process to get the solution $p^{(1)}$

$$p_n^{(1)''}(x) + k_n^2 p_n^{(1)}(x) = \sum_{m=0}^{N+1} D_{nm}(x)p_m^{(0)}(x) + E_{nm}(x)p_m^{(0)'}(x). \tag{1.9}$$

Applications of the first Born approximation are presented in section 4, leading to improved approximate equations compared to the usual Webster approximation.

The main advantage of system (1.8) is that, in the straight part of the waveguide, we get $p_n'' + k_n^2 p_n = 0$, for $n \leq N$, as expected. More surprisingly, the equation for $n = N+1$ is:

$$p_{N+1}'' + K^2 p_{N+1} = 0,$$

and introduces a new wavenumber K (we prefer the notation K rather than k_{N+1} to avoid confusion with the usual wavenumber). Although this wavenumber is unphysical, it turns out that $K^2 < 0$, indicating that the boundary mode can be interpreted as an evanescent mode. Instead of imposing vanishing value of p_{N+1} at the inlet/outlet of the scattering region, as in Athanassoulis & Belibassakis (1999, 2003) and Hazard & Lunéville 2008, we can associate to the boundary mode a radiation condition $p_{N+1}' = \pm iKp_{N+1}$. This makes possible the implementation of efficient numerical multimodal methods, as proposed in Pagneux *et al.* 1996, Felix & Pagneux 2001, Pagneux & Maurel 2002.

The paper is organized as follows. The Section 2 presents in detail the derivation of the improved modal system written using the modal components (p_n) for the case of a waveguide with one varying wall. The advantages of this formulation are discussed when compared to the classical formulation (without boundary mode). It is shown that thanks to the boundary mode, the system can be practically truncated at the number of propagating modes. The boundary condition on the varying wall is also inspected as a function of the truncation number, leading to the conclusion that the desired behavior of the truncated solution $p^N(x,y)$, namely $\partial_y p^N(x,h) = h'(x)\partial_x p^N(x,h)$, is reached asymptotically, for $N \to \infty$. The case of two varying walls, where two boundary modes are considered, is presented in Section 3. Although more involved, the conclusions are the same in this case. Finally, for low frequencies, the systems can be solved in the first Born approximation. The contribution of the boundary mode(s) to the plane wave is shown to improve the Webster equation. This is considered in the Section 4 and illustrated with examples. Technical calculations and discussion on the derivation of the modal systems are collected in the Appendices.

2. The case of one varying wall

(*a*) *Derivation of the improved coupled representation*

The initial geometry of the waveguide is set in the (x,y) plane with $y \in [0, h(x)]$ and $x \in]-\infty, \infty[$. Let us introduce a change of variables to get a problem set in a straight guide of unitary height. Considering the transformation $X = x, Y = y/h(x)$ and with $p(x,y)$ satisfying the Helmholtz equation (1.1), the field $p(X,Y)$ satisfies:

$$\nabla \cdot \left[\frac{J^T J}{\det(J)} \nabla p(X,Y) \right] + \frac{k^2}{\det(J)} p(X,Y) = 0,$$

where J is the Jacobian of the transformation $(x,y) \to (X,Y)$:

$$J = \begin{pmatrix} 1 & -h'Y/h \\ 0 & 1/h \end{pmatrix}.$$

We deduce the modified Helmholtz equation:

$$\partial_X (h \partial_X p - h'Y \partial_Y p) - \partial_Y \left(h'Y \partial_X p - \frac{1 + h'^2 Y^2}{h} \partial_Y p \right) + k^2 h p = 0, \tag{2.1}$$

for $X \in]-\infty, \infty[$, $Y \in]0,1[$. The boundary conditions $\partial_y p(x,0) = 0$ and (1.2) become:

$$\begin{cases} \partial_Y p(X,0) = 0, \\ h h' \partial_X p(X,1) = (1 + h'^2) \partial_Y p(X,1). \end{cases} \tag{2.2}$$

The equation (2.1) is projected onto a basis $\varphi_n(Y)$ and we get

$$\begin{aligned} &\int_0^1 dY \varphi_n \left[\partial_X (h \partial_X p - h'Y \partial_Y p) + k^2 h p \right] + \\ &+ \int_0^1 dY \varphi_n' \left[h'Y \partial_X p - \frac{1 + h'^2 Y^2}{h} \partial_Y p \right] = 0, \end{aligned} \tag{2.3}$$

where we have used the boundary conditions Eq. (2.2). The system of coupled mode equations resulting from this projection depends on the expansion chosen for p. First, we will derive the system of equations using the usual expansion without boundary mode. Then, we will show that using the expansion as used in Athanassoulis & Belibassakis 1999 and in Hazard & Lunéville 2008 results in a system of equations that remain coupled in the straight parts of the waveguide (where $h'(X)=0$). As already mentioned this is not suitable to use the Born approximation or to define radiation conditions. Finally, we use our new formulation based on a similar expansion, leading to a system that decouples the modes in the straight parts of the waveguide.

(a.1) *Classical representation*

In the classical modal approach (without additional boundary mode):

$$p(X,Y)=\sum_{n=0}^{N} p_n(X)\varphi_n(Y),$$

where $\varphi_n(Y)$ are the transverse modes of a straight unitary waveguide:

$$\begin{cases} \varphi_0(Y)=1, \\ \varphi_n(Y)=\sqrt{2}\cos(n\pi Y). \end{cases}$$

Because of the orthogonality of the φ_n-functions, $(\varphi_n,\varphi_m)=\delta_{mn}$, the functions p_n are the modal components $p_n(X)\equiv(p,\varphi_n)=\int_0^1 p(X,Y)\varphi_n(Y)dY$. We get, from Eq. (2.3) and for $0\le n\le N$:

$$(hp_n')'+k_n^2 hp_n = \sum_{m=0}^{N}\left[\frac{h'^2}{h}d_{mn}p_m - h'a_{mn}p_m' + a_{nm}(h'p_m)'\right], \tag{2.4}$$

where we have used $(\varphi_n',\varphi_m')=(n\pi)^2\delta_{nm}$ to get $k_n(x)^2\equiv k^2-(n\pi/h(x))^2$ and where we have defined:

$$a_{mn}\equiv(Y\varphi_m,\varphi_n'),\;\; d_{mn}\equiv(Y^2\varphi_m',\varphi_n'). \tag{2.5}$$

The above system differs from the derivation proposed in Pagneux *et al.* 1996 (this derivation corrects an error in the original derivation of Stevenson 1951). This is discussed in Appendix 1. As expected in the straight parts of the waveguide we recover the usual 1D Helmholtz equations $p_n''+k_n^2p_n=0$.

(a.2) *Coupled mode equations for the q_n-components*

We consider now the expansion of p as done in Athanassoulis & Belibassakis 1999:

$$p(X,Y)=\sum_{n=0}^{N} q_n(X)\varphi_n(Y)+q_{N+1}(X)\chi(Y),$$

and we choose χ such as $\chi'(1)\neq 0$. For instance, a convenient choice is:

$$\chi(Y)=\sqrt{2}\cos(\pi Y/2).$$

Note that $(\varphi_0, \cdots \varphi_N)$ and χ are linearly independent as soon as N is finite. The system of coupled mode equations can be found in Hazard & Lunéville 2008 (see Eq. (4.11) in this reference). Here, we just inspect the form of this system in the straight parts of the waveguide. We get, for $h'(X) = 0$:

$$\begin{cases} (q_n'' + \alpha_n q_{N+1}'') + k_n^2(q_n + \alpha_n q_{N+1}) = 0, & \text{for } 0 \leq n \leq N, \\ \displaystyle\sum_{n=0}^{N} \alpha_n(q_n'' + k_n^2 q_n) + q_{N+1}'' + \left[k^2 - \left(\frac{\pi}{2h}\right)^2\right] q_{N+1} = 0, & \end{cases} \tag{2.6}$$

with

$$\alpha_n \equiv (\chi, \varphi_n), \quad n \leq N.$$

Surprisingly, the q_n components, $n \leq N$, appear to be coupled with the q_{N+1} component, although we expect q_{N+1} to be useless in the straight parts of the guide. In Athanassoulis & Belibassakis 1999, Hazard & Lunéville 2008, $q_{N+1} = 0$ is imposed outside a bounded calculation domain. Then (2.6) implies $q_n'' + k_n^2 q_n = 0$ for $n \leq N$ and the usual radiation conditions $q_n' = \pm i k_n q_n$ $(0 \leq n \leq N)$ for $\pm x \to \infty$ can be applied. Obviously, this means that the solutions $(q_n)_{n \leq N+1}$ depend on the size of the bounded calculation domain. In the next section, we show that it is possible to decouple the mode equations outside the scattering region independently of the size of the calculation domain. In the process the mode q_{N+1} is identified as an evanescent mode, associated to a new wavenumber K. Our coupled mode equation does not restrict to a bounded calculation domain, since a natural radiation condition $p_{N+1} = \pm i K p_{N+1}$ is obtained.

(a.3) *New coupled mode equations on the p_n-components*

To prevent the modal components to remain coupled in the straight parts of the waveguide, we use a reformulation of the expansion in terms of new unknows p_n such that:

$$p(X, Y) = \sum_{n=0}^{N} p_n(X)\varphi_n(Y) + p_{N+1}(X)\varphi_{N+1}(Y), \tag{2.7}$$

with

$$\varphi_{N+1}(Y) \equiv \chi(Y) - \sum_{n=0}^{N} \alpha_n \varphi_n(Y). \tag{2.8}$$

The p_n and the q_n components are linked by the relations:

$$p_n(X) \equiv (p, \varphi_n) = \begin{cases} q_n(X) + \alpha_n q_{N+1}(X), & \text{if} \quad 0 \leq n \leq N, \\ q_{N+1}(X), & \text{if} \quad n = N+1. \end{cases}$$

We get, for $0 \leq n, m \leq N+1$,

$$\beta_n (h p_n')' + \sum_{m=0}^{N+1} a_{mn} h' p_m' - a_{nm}(h' p_m)' - \frac{1}{h}\left[\gamma_n \delta_{mn} - k^2 h^2 \beta_n \delta_{mn} + d_{mn} h'^2\right] p_m = 0, \tag{2.9}$$

where a_{nm} and d_{nm} are defined in Eq. (2.5) (and applied here for $0 \le n, m \le N+1$) and with

$$(\varphi_m, \varphi_n) \equiv \beta_n \delta_{mn}, \ (\varphi'_m, \varphi'_n) \equiv \gamma_n \delta_{mn}. \tag{2.10}$$

The above properties are important. The functions $(\varphi_n)_{n \le N+1}$ are orthogonal, by construction and the functions $(\varphi'_n)_{n \le N+1}$ are also orthogonal. It follows that our reformulation of the modal expansion succeeds in decoupling the modal components in the straight parts of the waveguide: for the $N+1$ first modes, $0 \le n \le N$, we recover the expected propagation equations $p''_n + k_n^2 p_n = 0$, as in the classical projection. For $n = N+1$, we get

$$p''_{N+1} + \left[k^2 - \frac{1}{h^2}\frac{\gamma_{N+1}}{\beta_{N+1}}\right] p_{N+1} = 0. \tag{2.11}$$

We have $(\beta_n)_{n \le N} = 1$, $(\gamma_n)_{n \le N} = (n\pi)^2$ and

$$\begin{cases} \beta_{N+1} = 1 - \displaystyle\sum_{n=0}^{N} \alpha_n^2 \sim \frac{1}{3\pi^2}\frac{1}{N^3}, \\ \gamma_{N+1} = \left(\dfrac{\pi}{2}\right)^2 - \displaystyle\sum_{n=0}^{N} (n\pi)^2 \alpha_n^2 \sim \frac{1}{N}. \end{cases} \tag{2.12}$$

This shows that the additional mode is associated to a wavenumber K such that:

$$K^2 \equiv \left[k^2 - \frac{1}{h^2}\frac{\gamma_{N+1}}{\beta_{N+1}}\right], \tag{2.13}$$

Let us prove now that p_{N+1} is an evanescent mode. For each x value we note $n_p(x)$ the number of propagating modes (n_p is the integer part of $kh(x)/\pi$ plus one). If we introduce $N_p = \sup n_p(x)$ where the sup is taken on all the x values, the mode φ_{N_p} is the first mode evanescent in the whole waveguide. Assuming that the truncation does not eliminate the propagating modes ($N \ge N_p - 1$), we have $K^2 < 0$ which corresponds to an evanescent mode. Indeed, using $n\pi\alpha_n \ge (N+1)\pi\alpha_n$ for all $n \ge N+1$, we get:

$$K(x)^2 = k^2 - \frac{1}{h(x)^2} \frac{\displaystyle\sum_{n=N+1}^{\infty} (n\pi\alpha_n)^2}{\displaystyle\sum_{n=N+1}^{\infty} \alpha_n^2} \le k^2 - \left(\frac{(N+1)\pi}{h(x)}\right)^2 \le k^2 - \left(\frac{N_p\pi}{h(x)}\right)^2 = k_{N_p}^2,$$

and $k_{N_p}^2 < 0$ for all x values by definition of N_p.

We come back to our system of coupled mode equations. In the part of the waveguide with varying cross section, we have

$$p''_n + \left[k^2 - \frac{1}{h^2}\frac{\gamma_n}{\beta_n}\right] p_n = \frac{1}{\beta_n} \sum_{m=0}^{N+1} \left[a_{nm}\frac{1}{h}(h'p_m)' - (a_{mn} + \beta_n\delta_{mn})\frac{h'}{h}p'_m + \frac{h'^2}{h^2} d_{mn} p_m\right], \tag{2.14}$$

A reasonable question is what happens for $N \to \infty$. The equations for p_n, $n \le N+1$, involve $(a_{n,N+1},\, a_{N+1,n})$ and $(d_{N+1,n}, d_{n,N+1})$. We have, for $n \le N$

$$\left\{\begin{aligned} a_{n,N+1} &= (Y\varphi_n, \chi') - \sum_{m=0}^{N} \alpha_m a_{nm} = \sum_{m=N+1}^{\infty} \alpha_m a_{nm}, \\ a_{N+1,n} &= (Y\chi, \varphi_n') - \sum_{m=0}^{N} \alpha_m a_{mn} = \sum_{m=N+1}^{\infty} \alpha_m a_{mn}, \\ d_{N+1,n} &\equiv (Y^2\chi', \varphi_n') - \sum_{m=0}^{N} \alpha_m d_{mn} = \sum_{m=N+1}^{\infty} \alpha_m d_{mn}, \end{aligned}\right. \tag{2.15}$$

and

$$\left\{\begin{aligned} a_{N+1,N+1} &= \sum_{m=N+1}^{\infty} \alpha_m \alpha_n a_{nm}, \\ d_{N+1,N+1} &= \sum_{m=N+1}^{\infty} \alpha_m \alpha_n d_{mn}, \end{aligned}\right. \tag{2.16}$$

The expressions of the coefficients a_{nm} and d_{nm} are given in the Appendix 1. When $N \to \infty$, it is easy to check the following behaviors:

$$\left\{\begin{aligned} & d_{N+1,n} \sim \frac{8n^2(-1)^n}{3\pi}\frac{1}{N^3}, && a_{N+1,n} \sim \frac{2n^2(-1)^n}{3\pi}\frac{1}{N^3}, \\ & a_{0,N+1} \sim -\frac{\sqrt{2}}{\pi}\frac{1}{N}, && a_{n,N+1} \sim -\frac{2(-1)^n}{\pi}\frac{1}{N}, \\ & d_{N+1,N+1} \sim \frac{1}{N}, && a_{N+1,N+1} \sim \frac{1}{3\pi^2}\frac{1}{N^3}. \end{aligned}\right. \tag{2.17}$$

(an example of derivation of this equivalent is given in Appendix 1). Let us consider the equation on p_n, $n \le N$ in Eq. (2.14). It is easy to check that the equation tends to the classical equation Eq. (2.4) when $N \to \infty$ (without boundary mode). For instance $a_{nm} \sim m^2/(n^2 - m^2)$, so that $a_{nm} \to -1$ for large m, while $a_{n,N+1} \sim 1/N \to 0$. This is expected since the additional degree of freedom p_{N+1} becomes unnecessary. The equation on p_{N+1} has the same structure, but with an unphysical wavenumber K that depends on the truncation. Asymptotically, Eq. (2.14) for $n = N+1$ leads to constant p_{N+1} (the dominant term is $(h'/h)\delta_{m,N+1}\, p_m'$ in the right hand side term, which is $O(N^3)$) and uncoupled to the other modes p_m.

(*b*) *Convergence and boundary condition in the improved representation*

The convergence and the errors of the improved method compared to the classical method are illustrated on the Figs. 1 and 2. In the calculation, the non uniform part of the waveguide is given by $h(x) = h[1 + 0.75(1 + \cos 2\pi x)]$ for $x \in [-0.5, 0.5]$ (geometry A). A plane wave is sent from the left at a frequency $kh = 2$, for which $N_p = 2$, two propagating modes exist in the largest part of the waveguide. The presented results have been obtained using a numerical scheme based on the use of the admittance matrix, as described in Pagneux *et al.* 1996, Maurel *et al.* 2013, implementing the evanescent boundary mode like other evanescent modes.

The Figs 1 show the real part of the acoustic field, in the improved method for a truncation just at the two propagating modes ($N=1$, the field is described by $N_{dof}=3$ degrees of freedom, the two propagating modes and the boundary mode) and in the classical method for truncations including the two propagating modes plus 1 to 41 additional evanescent modes. It can be seen that the upper wall boundary condition is reasonably verified in the improved method while the classical method suffers from oscillations and it needs about 40 evanescent modes to behave satisfactorily at the walls. More quantitatively, the Fig. 2(a) shows the errors on the total field as a function of the truncation N. The reference field has been calculated considering $N=100$, for which the fields obtained with or without boundary modes are equal up to less than 1%. The error is defined in L^2 norm: $||p^N - p^{100}||_{L^2}/||p^{100}||_{L^2}$. The general trends of the errors are in agreement with the prediction of Hazard & Lunéville 2008, with an error varying like $N^{-3/2}$ in the classical method and $N^{-7/2}$ in the improved method (the dependence of the $||q_n||_{L^2}$ and $||p_n||_{L^2}$ modal components are also reported on Fig. 2(b) as a function of n; $||q_n||_{L^2} \propto n^{-4}$ in the improved method while $||p_n||_{L^2} \propto n^{-2}$ in the classical modal method). More remarkable is the case where $N_{dof}=3$ degrees of freedom (3 terms taken into account in the modal expansion) are considered, already seen qualitatively on Figs. 1. Adding to the 2 propagating modes, the third degree of freedom is the first evanescent mode in the classical method and it is the boundary mode in the improved method. The resulting error on the total field is about 30% in the classical method while it is only 5% in the improved method, and one has to consider more than 10 evanescent modes in the classical method to reach the same small error (note that, although the error on the total field is small for N around 10, one can see on Fig. 1 that the field still suffers from oscillations).

We now focus on the boundary condition on the non uniform wall at $y=h(x)$. The mode p_{N+1} has been shown to be an evanescent mode. This mode is excited in the perturbed region $h' \neq 0$ but is not equal to zero outside this region, it is only exponentially decreasing in the straight part of the waveguide. This seems to be in contradiction with the desired condition $\partial_y p(x,h) = h'\partial_x p(x,h) = 0$ in the straight part. To clarify this point, let us consider the truncated condition as a function of the truncation number. The change of variable $(x,y) \to (X,Y)$ implies that:

$$\begin{aligned} \partial_y p^N(x,h) - h'\partial_x p^N(x,h) &= \frac{1+h'^2}{h}\partial_Y p^N(X,1) - h'\partial_X p^N(X,1) \\ &= \frac{1+h'^2}{h} p_{N+1}(x)\chi'(1) - h'\sum_{m=0}^{N} q'_m(x)\varphi_m(1), \end{aligned} \tag{2.18}$$

and we will prove now that this quantity is of order $O(1/N)$. In this aim the equation for p_{N+1} in (2.14) is rearranged in the following way:

$$\begin{aligned} &\frac{\gamma_{N+1} + h'^2 d_{N+1,N+1}}{h} p_{N+1} - h'\sum_{n=0}^{N} a_{n,N+1} p'_n = \beta_{N+1}\left[(hp'_{N+1})' + k^2 h p_{N+1}\right] \\ &+ a_{N+1,N+1}\left[h'p'_{N+1} - (h'p_{N+1})'\right] - \sum_{n=0}^{N}\left[\frac{h'^2}{h} d_{n,N+1} p_n + a_{N+1,n}(h'p_n)'\right]. \end{aligned} \tag{2.19}$$

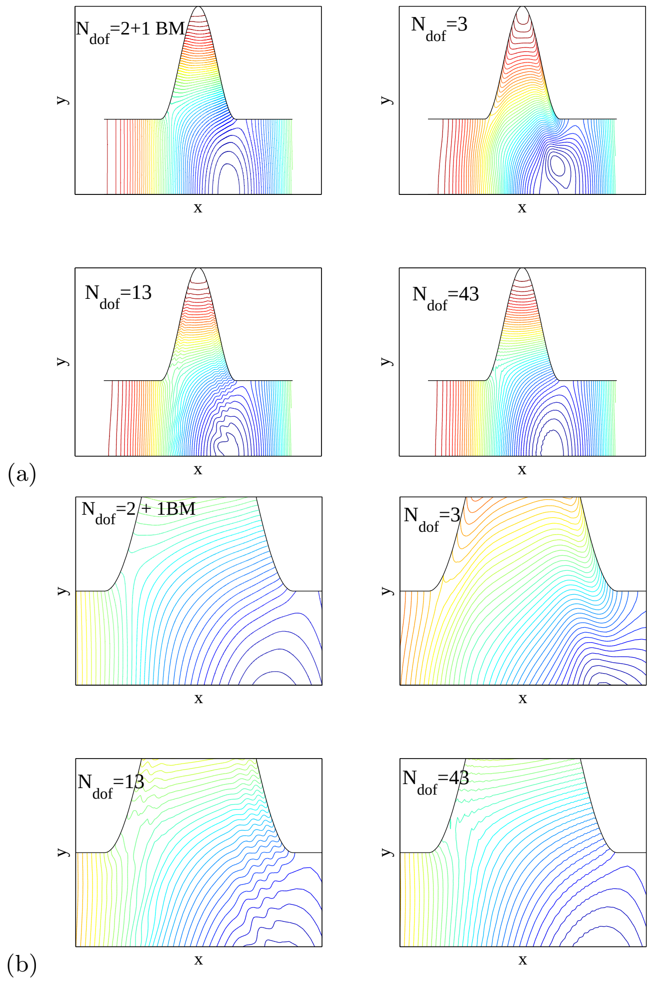

Figure 1. Real part of the acoustic field in the geometry A, with $kh = 2$ (two propagating modes for $kh_{max} = 5$). With the boundary mode and after truncation at the two propagating modes ($N_{dof} = 2 + 1\text{BM}$), and without boundary mode after truncation at $N = N_{dof} = 3$, 13 and 43. The set of figures (b) on the right shows zooms of the figures (a) on the left.

Owing to the behaviour of the coefficients in Eqs. (2.12) and (2.17), it appears that the left handside in (2.19) is of order $1/N$ (let us recall that p_n decreases with n, $|p_n| \sim 1/n^2$) whereas the right handside is of order $1/N^2$. More precisely we get at dominant order:

$$\frac{1+h'^2}{h} p_{N+1}(X) + h' \frac{1}{\pi} \left[\sqrt{2} p_0'(X) + 2 \sum_{n=0}^{N} (-1)^n p_n'(X) \right] = O(1/N).$$

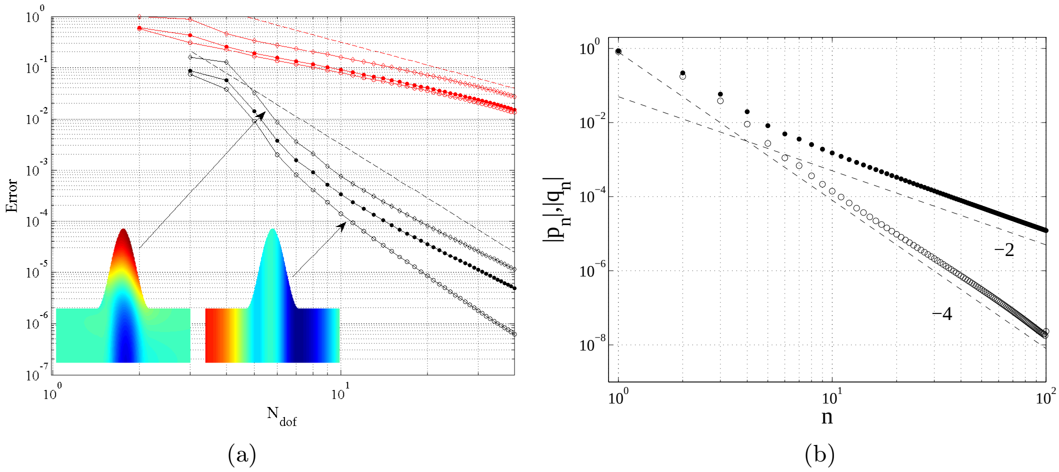

Figure 2. (a) Errors as a function of the truncation in the geometry A. Red symbols for the classical method and black symbols for the improved method; open circles for the error in the propagating part of the field, diamonds for the errors for the evanescent part of the field and close circles for the error on the total fields. The insets show the evanescent part of the field (left inset) and the propagating part of the field (right inset). (b) Dependence on q_n (open symbols) and p_n (closed symbols) as a function of n in the geometry A, dotted lines are guidelines with slopes -2 and -4.

Using $\varphi_0(1)=1$, $\varphi_n(1)=\sqrt{2}(-1)^n$, $0<n\leq N$ and $\chi'_{N+1}(1)=-\pi/\sqrt{2}$, the above equation can be written:

$$\frac{1+h'^2}{h}p_{N+1}(X)\chi'(1)-h'\sum_{m=0}^{N}p'_m(x)\varphi_m(1)=O(1/N).$$

It is also easy to see, from $p_n=q_n+\alpha_n p_{N+1}$, that

$$\sum_{n=0}^{N}q'_n(x)\varphi_n(1)=\sum_{n=0}^{N}p'_n(x)\varphi_n(1)+O(1/N),$$

(using $\sum_{n=1}^{N}1/(n^2-1/4)=2+O(1/N)$). It follows that the truncated boundary condition (2.18) satisfies:

$$h'\partial_x p^N(x,h)-\partial_y p^N(x,h)=O(1/N),$$

and the right boundary condition is verified for $N\to\infty$.

3. The case of two varying walls

In the case of two varying walls of shapes $h_1(x)$ and $h_2(x)$, two extra degrees of freedom $p_{N+1}(x)$ and $p_{N+2}(x)$ are considered in the expansion of $p(x,y)$, and it is expected that each degree of freedom will tackle one of the two boundary conditions on the non uniform walls.

The procedure to derive a family of 1D wave equations is similar to the case of one varying wall. The transformation $(x, y) \to (X, Y)$ is now defined by $x = X$ and $y = h_1 + Yh$ where $h = h_2 - h_1$. The Jacobian of this transformation is:

$$J = \begin{pmatrix} 1 & -f(Y)/h \\ 0 & 1/h \end{pmatrix},$$

with $f(Y) \equiv h_1' + Yh'$, from which we deduce the modified Helmhotlz equation:

$$\partial_X(h\partial_X p - f\partial_Y p) - \partial_Y\left(f(Y)\partial_X p - \frac{1+f^2}{h}\,\partial_Y p\right) + k^2hp = 0,$$

with boundary conditions:

$$\begin{cases} hf(0)\partial_X p(X,1) = \left[1 + f(0)^2\right]\partial_Y p(X,0), \\ hf(1)\partial_X p(X,1) = \left[1 + f(1)^2\right]\partial_Y p(X,1). \end{cases} \tag{3.1}$$

$p(X, Y)$ is expanded onto the usual basis $(\varphi_n)_{n\geq 0}$ and on two additional modes φ_{N+1} and φ_{N+2}:

$$p(X,Y) = \sum_{n=0}^{N} p_n(X)\varphi_n(Y) + p_{N+1}(X)\varphi_{N+1}(Y) + p_{N+2}(X)\varphi_{N+2}(Y),$$

and we choose φ_{N+1} and φ_{N+2} as :

$$\begin{cases} \varphi_{N+1}(Y) \equiv \chi_1(Y) - \displaystyle\sum_{n=0}^{N} \alpha_n^{(1)}\varphi_n(Y), \\ \varphi_{N+2}(Y) \equiv \chi_2(Y) - \displaystyle\sum_{n=0}^{N} \alpha_n^{(2)}\varphi_n(Y), \end{cases}$$

with

$$\begin{cases} \chi_1(Y) \equiv \xi_1\left[\cos(\pi Y/2) + \sin(\pi Y/2)\right], \\ \chi_2(Y) \equiv \xi_2\left[\cos(\pi Y/2) - \sin(\pi Y/2)\right]. \end{cases}$$

and

$$\alpha_n^{(j)} \equiv (\chi_j, \varphi_n), \quad j = 1, 2. \tag{3.2}$$

The normalizations are $\xi_1 = 1/\sqrt{1 + 2/\pi}$ and $\xi_2 = 1/\sqrt{1 - 2/\pi}$. The projection of the wave equation (3.1) is

$$\begin{aligned} \beta_n(hp_n')' \quad & - \sum_{m=0}^{N+2}\left[\left(a_{nm}h' + h_1'b_{nm}\right)p_m'\right]' + \left[a_{mn}h' + h_1'b_{mn}\right]p_m' \\ & - \sum_{m=0}^{N+2}\frac{1}{h}\left[(1 + h_1'^2)\gamma_n\delta_{mn} - k^2h^2\beta_n\delta_{mn} + d_{mn}h'^2 + 2h'h_1'c_{mn}\right]p_m = 0, \end{aligned}$$

for $0 \leq n, m \leq N + 2$, with a_{nm} and d_{nm} defined in Eq. (2.5) and

$$b_{mn} \equiv (\varphi_m, \varphi_n'), \;\; c_{mn} \equiv (Y\varphi_m', \varphi_n').$$

The key point is that we have chosen χ_1 and χ_2 such that $(\varphi_{N+1}, \varphi_{N+2}) = (\varphi_{N+1}', \varphi_{N+2}') = 0$ (otherwise, for $n \leq N$, $(\varphi_{N+1}, \varphi_n) = (\varphi_{N+2}, \varphi_n) = 0$ and

$(\varphi'_{N+1},\varphi'_n)=(\varphi'_{N+2},\varphi'_n)=0$ by construction). This is because we have chosen $(\chi_1,\chi_2)=(\chi'_1,\chi'_2)=0$ and such that $\alpha_n^{(1)}\alpha_n^{(2)}=0$ for any n-value (see Eqs. (A.10)), from which we deduce

$$\begin{aligned}(\varphi_{N+1},\varphi_{N+2}) &= (\chi_1,\chi_2)-\sum_{n=0}^{N}\alpha_n^{(1)}\alpha_n^{(2)}=0,\\ (\varphi'_{N+1},\varphi'_{N+2}) &= (\chi'_1,\chi'_2)-\sum_{n=0}^{N}\alpha_n^{(1)}\alpha_n^{(2)}(n\pi)^2=0,\end{aligned} \tag{3.3}$$

Again, the equations on p_n are decoupled in the straight parts of the waveguide, with $p''_n+k_n^2p_n=0$ (and $k_n^2=k^2-(n\pi/h)^2$) for the usual modes $n\le N$ and

$$p''_{N+j}+K_j^2p_{N+j}=0,\ \ j=1,2, \tag{3.4}$$

and

$$K_j^2\equiv\left[k^2-\frac{1}{h^2}\frac{\gamma_{N+j}}{\beta_{N+j}}\right]. \tag{3.5}$$

In the part of the waveguide with varying cross section, we have, for $n=0,\dots(N+2)$

$$\begin{aligned}p''_n+\ \left[k^2-\frac{1}{h^2}\frac{\gamma_n}{\beta_n}\right]p_n &= \frac{1}{\beta_n}\sum_{m=0}^{N+2}\left[\frac{1}{h}\left(a_{nm}h'+b_{nm}h'_1\right)p'_m\right]'\\ &-\left[(a_{mn}+\beta_n\delta_{mn})\frac{h'}{h}+b_{mn}\frac{h'_1}{h}\right]p'_m+\left[\gamma_n\delta_{mn}\frac{h_1'^2}{h^2}+d_{mn}\frac{h'^2}{h^2}+2c_{mn}\frac{h'h'_1}{h^2}\right]p_m,\end{aligned} \tag{3.6}$$

The coefficients $(a_{nm},b_{nm},c_{nm},d_{nm})$ and (γ_n,β_n) are given in the Appendix 1. As for one varying wall, we find that the modal components $p_{n\le N}$ are the usual modal components associated to the wavenumbers $k_n=\sqrt{k^2-(n\pi/h(x))^2}$, while the two boundary modes appear to be evanescent with purely imaginary wavenumbers $K_j\equiv\sqrt{k^2-\gamma_{N+j}/\beta_{N+j}h^2}$, $j=1,2$.

Typical results are shown on Figs. 3 for two varying walls with $h_1(x)=(\tan\alpha)x$ for $x\in[0,1]$ and $h_2(x)=h_1(x)+1$. We consider $\alpha=0.25\pi$ (geometry B) and $\alpha=0.37\pi$ (geometry C). At a frequency $kh=2$, only the plane mode is propagating. It can be seen that the boundary conditions on each walls are reasonably verified as soon as $N=2$ in the improved method ($N_{dof}=4$) while the classical method suffers from oscillations after truncation at a few modes and it needs more than 10 evanescent modes to approach reasonably the boundary conditions. The conclusions (not reported) on the convergence versus N and the boundary conditions are the same as in the case of one varying wall.

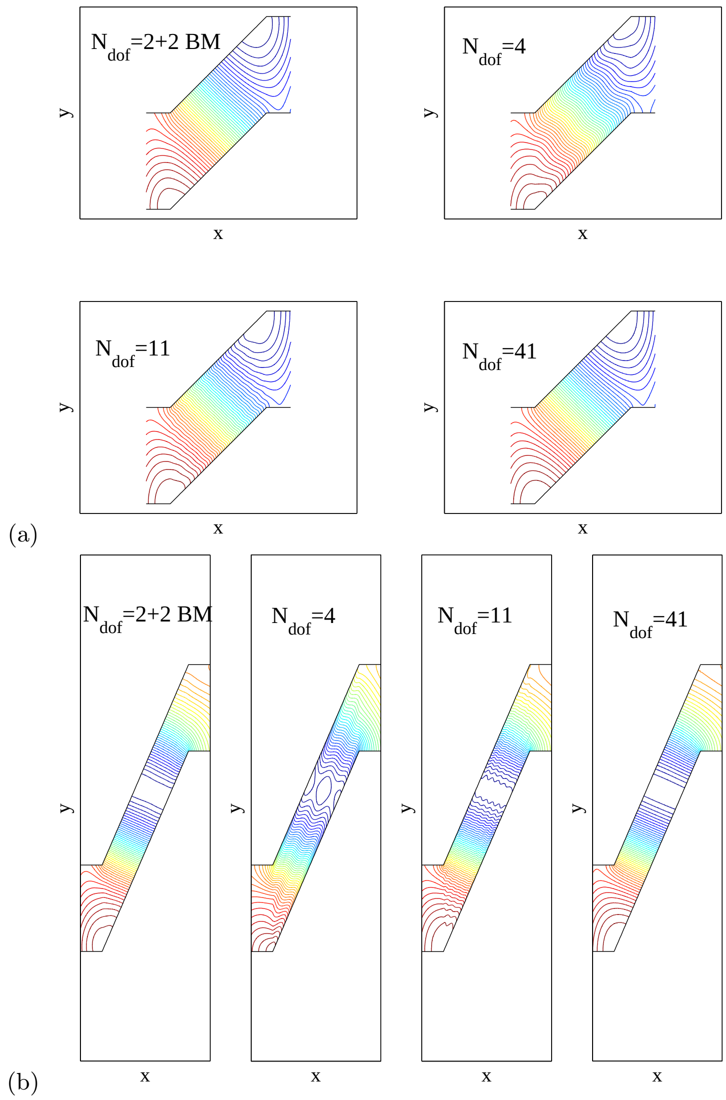

Figure 3. Real part of the acoustic field in the geometry B (left) and C (right), with $kh = 2$. With the boundary modes and after truncation at the two propagating modes ($N_{dof} = 2 + 2$BM), and without boundary mode after truncation at $N = N_{dof} = 4$, 11 and 41.

4. Low frequency limit: Revisiting Webster equation

In this section, the low frequency regime is inspected and the possibility to improve the usual Webster approximation, namely

$$h(p_0'' + k^2 p_0) = -h' p_0', \tag{4.1}$$

where $p_0(x) = e^{ikx}$ at leading order is investigated. At a given frequency kh (h is the height outside the perturbed area), the Webster equation inspects the first order correction in $O(h')$ due to the inhomogeneity in the cross section. A derivation of the Webster equation can be found in Rienstra 2005 (see also a brief discussion in the Appendix 1).

We will show that, (2.14) for one boundary mode and (3.6) for two boundary modes degenerate in coupled equations of the general form

$$\begin{cases} h(p_0'' + k^2 p_0) = -h' p_0' + G\left(h'\tilde{p}\right)', \\ h(\tilde{p}'' + K^2 \tilde{p}) = -F h' p_0', \end{cases} \tag{4.2}$$

where $K, \tilde{p}$ refer to the wavenumber and modal component of one boundary mode, F, G are constants that will be given latter. The coupled system involves now $\tilde{p} = O(h')$ so that the plane mode p_0 can be determined up to $O(h'^2)$. In the following, we refer to the Improved Webster Approximation as IWA (note that a different improvement is proposed for axisymmetric 3D geometry in Martin 2004).

Although Webster's equation (4.1) and the improved representation (4.2) can be solved numerically, we consider here analytical solutions by using the first Born approximation (namely $p_0(x) = e^{ikx}$ at leading order). We inspect two cases leading to simplifications. In the first case, $h(x)$ varies on a typical scale L much larger than h (slowly varying section) and in the second case L is much smaller than any scale of the problem (sudden variation).

(*a*) *General expressions*

In this section, we propose an analytical solution of the IWA Eq. (4.2), considering the case of one boundary mode. The case of two boundary modes follows simply by linearity. The second equation of (4.2) is solved in the first Born approximation ($p_0(x) \simeq e^{ikx}$):

$$\tilde{p}(x) = -ikF \int dy\, g_K(x-y)\, \frac{h'(y)}{h(y)} e^{iky}, \tag{4.3}$$

where $g_K(x) \equiv e^{iK|x|}/(2iK)$ is the Green function for the 1D wave equation. Reporting the solution for $\tilde{p}$ in the first equation of (4.2) leads to:

$$p_0(x) = e^{ikx} + p_0^W + \frac{G}{h} \int dy\, g_k'(x-y)\, h'(y) \tilde{p}(y), \tag{4.4}$$

where p_0^W refers to the solution of the Webster equation (4.1) and where $g_k'(x-y)$ denotes the derivative with respect to x. Note that we have used $h(x) \simeq h$ (remember that $|h'| \ll 1$) in the second equation.

(a.1) *Slowly varying waveguide*

We consider a cross section varying of $\Delta h \ll h$ over a length L with $h/L \ll 1$. As previously said, K is imaginary, and here, $K^2 = k^2 - \gamma_{N+1}/\beta_{N+1} h^2 \simeq -\sigma^2/h^2$, with $\sigma^2 \equiv \gamma_{N+1}/\beta_{N+1}$. We use $\int dy\, e^{-|y|/l} f(y) \sim 2l f(0)$ for f varying over a typical length $L \gg l$. Applying this result with $\sigma L/h \gg 1$, we get from Eq. (4.3), at dominant order and in the perturbed area (where $h' \neq 0$):

$$\tilde{p}(x) = ikh\frac{F}{\sigma^2}h'(x)e^{ikx}. \tag{4.5}$$

The equation (4.4) simply becomes

$$p_0(x) = e^{ikx} + p_0^W + ikC\int dy\, h'(y)^2 \mathrm{sgn}(x-y)e^{ik|x-y|+iky}, \tag{4.6}$$

where $C \equiv FG/(2\sigma^2)$.

(a.2) *Sudden variation in the section*

We consider now a sudden variation located at $x = x_0$, in which h varies of Δh on a typical scale L much smaller than any scale of the problem. Then $h'(x) = \Delta h\, \delta(x - x_0)$ and Eq.(4.3) reduces to:

$$\tilde{p}(x) = \frac{ik}{2}\frac{F}{\sigma}\Delta h e^{ikx_0}e^{-\sigma|x-x_0|/h}. \tag{4.7}$$

Then, the solution of the equation (4.4) simply follows (note that here, the solution p_0^W takes a very simple form since its second order in $(\Delta h/h)^2$ vanishes):

$$p_0(x) = e^{ikx} - \frac{\Delta h}{2h}\left[1 - 2iD(kh)\frac{\Delta h}{h}\,\mathrm{sign}(x-x_0)\right]e^{ikx_0}e^{ik|x-x_0|} + O\left(\left(\frac{\Delta h}{h}\right)^3\right), \tag{4.8}$$

with $D = FG/4\sigma$.

(*b*) *The case of one varying boundary*

Here the case of one boundary mode is treated and a discussion on the use of two boundary modes is proposed in the following section. With $a_{00} = d_{00} = 0$, and, for $N = 0$, we have from Eqs. (A.5)-(A.7): $a_{0+1,0} = d_{0,0+1} = 0$ and $a_{0,0+1} = -2\sqrt{2}/\pi$ (and $\gamma_{0+1} = (\pi/2)^2$, $\beta_{0+1} = 1 - 8/\pi^2$, from Eqs. (A.3)-(A.4)), the system (2.14) reduces to the following system at dominant order:

$$\begin{cases} h(p_0'' + k^2 p_0) = -h'p_0' + a_{0,0+1}\left[h'\tilde{p}\right]', \\ h(\tilde{p}'' + K^2\tilde{p}) = -\dfrac{a_{0,0+1}}{\beta_{0+1}}\, h'p_0', \end{cases}$$

that can be identified to the system (4.2) with $F = a_{0,0+1}/\beta_{0+1}$ and $G = a_{0,0+1}$. Then, the expressions (4.6) and (4.8) can be directly applied, taking $C = a_{0,0+1}^2/(2\gamma_{0+1}) \sim 0.1643$ and $D = a_{0,0+1}^2/(4\sqrt{\beta_{0+1}\gamma_{0+1}}) \sim 0.2964$. The comparisons of IWA solutions to the Webster Approximation WA are illustrated on Figs. 4 and 5 for a slowly varying waveguide with cosine modulation and for a waveguide with the upper boundary varying suddenly. In both cases, a significant improvement is observed although a quite high frequency $kh = 2$ is considered. In the case of the sudden variation of the upper boundary (Fig. 5), the IWA solution is able to capture the discontinuity of the plane mode at the expansion location (from the Webster equation (4.1) written in the conservative form $(hp_0')' + k^2 hp_0 = 0$ it is possible to prove that its solution is continuous, even for h discontinuous).

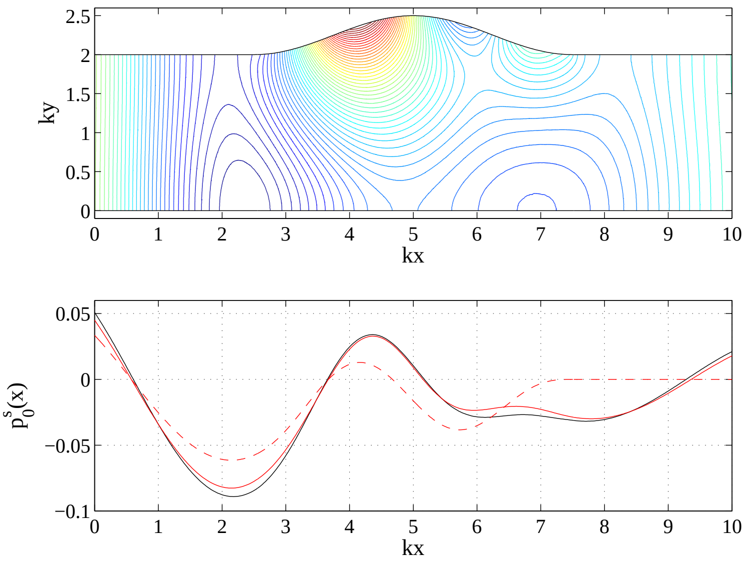

Figure 4. On the top: Real part of the scattered pressure field $p(x, y) - e^{ikx}$ in a waveguide with a cosine modulation of the upper boundary (localized between $kx = 2.5$ and $kx = 7.5$ with amplitude 0.5) at the frequency $kh = 2$. On the bottom: Real part of the plane mode $p_0(x) - e^{ikx}$, deduced from the reference calculation (black line), in the WA (doted red line) and in the IWA, including the contribution of the boundary mode (plain red line from Eq. (4.6)), with $C = 0.1643$.

(*c*) *The case of two varying walls*

We use $a_{00} = b_{00} = c_{00} = d_{00} = 0$, to get, from Eqs. (A.12)-(A.13): for $j = 1, 2$, $a_{0+j,0} = \hat{a}_0^{(j)} = 0$, $b_{0+j,0} = \hat{b}_0^{(j)} = 0$, $c_{0+j,0} = c_0^{(j)} = 0$, and $d_{0+j,0} = d_0^{(j)} = 0$; we also have $a_{0,0+1} = a_0^{(1)} = \xi_1(1 - 4/\pi)$, $b_{0,0+1} = b_0^{(1)} = 0$, $a_{0,0+2} = a_0^{(2)} = -\xi_2$, $b_{0,0+2} = b_0^{(2)} = -2\xi_2$, the system of equation (3.6) leads, at dominant order, to the simplified system for $(p_0, \tilde{p}_1, \tilde{p}_2)$:

$$\left\{ \begin{aligned} & h(p_0'' + k^2 p_0) = -h' p_0' + \left[\xi_1 (1 - 4/\pi) h' \tilde{p}_1 - \xi_2 (2h_1' + h') \tilde{p}_2 \right]', \\ & h(\tilde{p}_1'' + K_1^2 \tilde{p}_1) = -\frac{\xi_1}{\beta_1} (1 - 4/\pi) h' p_0', \\ & h(\tilde{p}_2'' + K_2^2 \tilde{p}_2) = \frac{\xi_2}{\beta_2} (2h_1' + h') p_0'. \end{aligned} \right.$$

By identifying the above system with Eqs. (4.2), it is straightforward to use Eqs. (4.6) and (4.8). Indeed, $\tilde{p}_1$ tackles the variation of cross section h with $F_1 \equiv \xi_1(1 - 4/\pi)/\beta_1$ and $G_1 = \xi_1(1 - 4/\pi)$ while $\tilde{p}_2$ tackles the variation of the centerline of the waveguide (here $2h_1 + h = h_1 + h_2$ plays the role of h' in (4.2)) with $F_2 =$

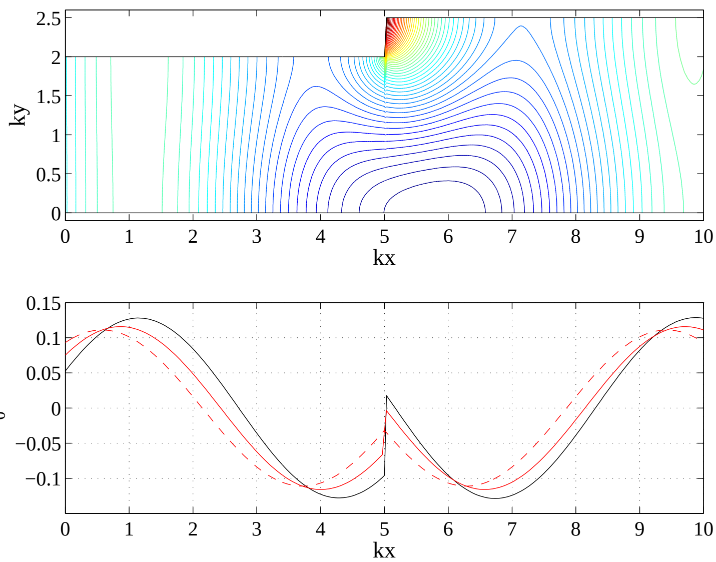

Figure 5. On the top: Real part of the scattered pressure field $p(x,y)-e^{ikx}$ in a waveguide with a sudden variation of the upper boundary (localized at $kx=5$ with amplitude 0.5) at the frequency $kh=2$. On the bottom: Real part of the plane mode $p_0(x)-e^{ikx}$, deduced from the reference calculation (black line), in the WA (doted red line) and in the IWA, including the contribution of the boundary mode (plain red line from Eq. (4.8)), with $D=0.2964$.

$-\xi_2/\beta_2$ and $G_2=-\xi_2$. We also have, from Eqs. (A.10)-(A.11), $\beta_1=1-(4\xi_1/\pi)^2$, $\gamma_1=(\pi/2)^2-\pi\xi_1^2$ and $\beta_2=1$, $\gamma_2=(\pi/2)^2+\pi\xi_2^2$.

More precisely, the Eqs. (4.5)-(4.6) becomes, for slowly varying h and h_1:

$$\left\{\begin{aligned} \tilde{p}_1(x)= &\quad ik\frac{\xi_1}{\gamma_1}\left(1-\frac{4}{\pi}\right)hh'(x)e^{ikx},\\ \tilde{p}_2(x)= &\quad -ik\frac{\xi_2}{\gamma_2}h[2h_1'(x)+h'(x)]e^{ikx},\\ p_0(x)= &\quad e^{ikx}+p_0^W(x)\\ &\quad +ik\int dy\,\left[C_1h'(y)^2+C_2\left[2h_1'(y)+h'(y)\right]^2\right]\mathrm{sign}(x-y)e^{ik|x-y|+iky}, \end{aligned}\right. \tag{4.9}$$

where $C_1\equiv\xi_1^2(1-4/\pi)^2/2\gamma_1=2(1-4/\pi)^2/[\pi^2(1-2/\pi)]\sim0.042$ and $C_2\equiv\xi_2^2/2\gamma_2=2/[\pi^2(1+2/\pi)]\sim0.124$. Note that in the case of one varying wall, $h_1'=0$, the solution for $p_0(x)$ in (4.9) is identical to the solution of Eq. (4.6) using one boundary mode by considering the change $C\to C_1+C_2$ and, as expected, we have $C_1+C_2\sim0.1655$, very close to the value $C\sim0.1643$ when using one boundary mode.

An illustration of a slowly varying waveguide is given in Fig. 6 for a constant section guide but varying h_1 (the deviation is sudden but h_1 is continuous). We considered the simplified expression obtained for a portion of waveguide between a and b with constant height h and forming an angle α with x (leading to $h' = 0$, $h'_1 = \tan\alpha$). Eq. (4.9) simplifies in:

$$p_0(x) = e^{ikx} + \frac{8ik}{\pi^2(1+2/\pi)}(\tan\alpha)^2 \int_a^b dy\, \text{sign}(x-y) e^{ik|x-y|+iky}. \tag{4.10}$$

The result is shown in Fig. 6 for $\alpha = 0.1$ and $(a = 2.5, b = 7.5)$ at a frequency $kh = 2$. Here, because $h' = 0$, the Webster's equation does not predict any effect of the bending on the plane mode and the effect captured by the boundary mode in IWA follows a $(\tan\alpha)^2$ law.

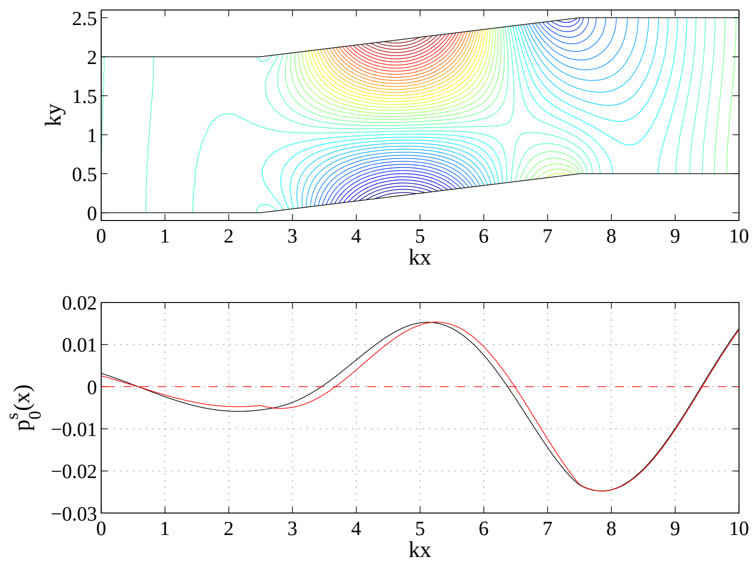

Figure 6. On the top: Real part of the scattered pressure field $p(x,y) - e^{ikx}$ in a waveguide with a sudden bend ($h'_1 = \tan\alpha = 0.1$ between $kx = 2.5$ and $kx = 7.5$) at the frequency $kh = 2$. On the bottom: Real part of the plane mode $p_0(x) - e^{ikx}$, deduced from the reference calculation (black line), in the WA (doted red line) and in the IWA, including the contribution of the boundary mode (plain red line from Eq. (4.10)).

If the variations are sudden for both walls, we consider $h_1(x) = \Delta h_1\, \delta(x - x_0)$ in addition to $h(x) = \Delta h\, \delta(x - x_0)$, which leads to, using Eqs. (4.7)-(4.8):

$$\begin{cases} \tilde{p}_1(x) = \dfrac{ik\xi_1}{2\beta_1\sigma_1}\left(1 - \dfrac{4}{\pi}\right)\Delta h\, e^{ikx_0} e^{-\sigma_1|x-x_0|/h}, \\ \tilde{p}_2(x) = -\dfrac{ik\xi_2}{2\beta_2\sigma_2}(2\Delta h_1 + \Delta h)\, e^{ikx_0} e^{-\sigma_2|x-x_0|/h}, \\ p_0(x) = e^{ikx} + \left\{-\dfrac{\Delta h}{2h} + ikh\, \text{sign}(x-x_0)\left[D_1\left(\dfrac{\Delta h}{h}\right)^2 + D_2\left(\dfrac{2\Delta h_1 + \Delta h}{h}\right)^2\right]\right\} e^{ikx_0} e^{ik|x-x_0|}. \end{cases} \tag{4.11}$$

where $D_1 = \xi_1^2(1-4/\pi)^2/(4\sqrt{\beta_1\gamma_1}) \simeq 0.1584$ and $D_2 \equiv \xi_2^2/(4\sqrt{\beta_2\gamma_2}) \simeq 0.2064$ (where we used $\sigma_j = \sqrt{\gamma_j/\beta_j}$, $j = 1, 2$). Note again that if only the upper boundary experiences a sudden change (Fig. 5 with $\Delta h_1 = 0$), the solution for $p_0(x)$ in (4.11) is identical to the solution of Eq. (4.8) using one boundary mode with $D \to D_1 + D_2$. Although we have here $D_1 + D_2 \sim 0.3648$, quite different from $D \sim 0.2964$ when using one boundary mode, no significant difference is observed in the IWA profile $p_0(x)$ (the result is not reported).

The result for a symmetric sudden expansion at frequency $kh = 2$ is shown in Fig. 7. The agreement between the reference solution and the IWA solution is excellent, better than the agreement found for a sudden change in the upper boundary only (Fig. 5). This is probably due to the symmetry of the sudden expansion: because odd modes are not allowed, the first evanescent mode which can be excited is the mode 2, associated to the cut off frequency $k_2h = 2\pi$. This frequency is far from the incident wave frequency $kh = 2$, which means that the mode 2 is very evanescent, well captured by the added boundary modes. On the contrary, for the non symmetric expansion, the mode 1 of cut off frequency $k_1h = \pi$ seems to be too close to a propagating mode to be satisfactorily tackled by the boundary mode.

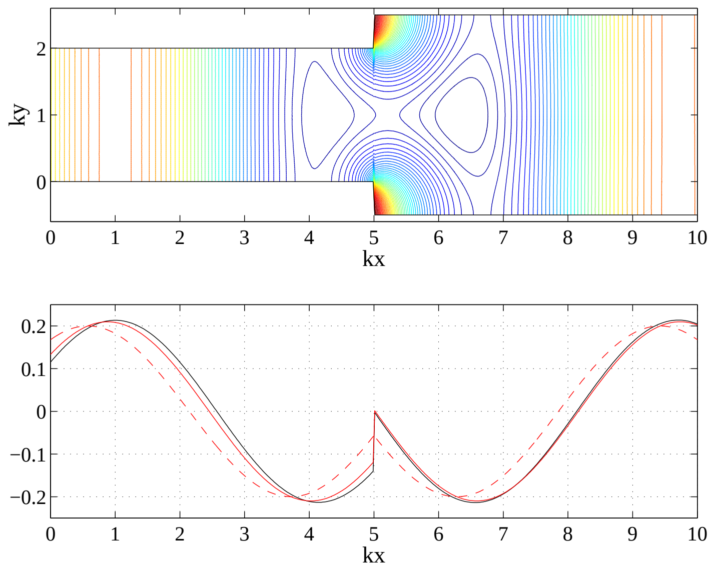

Figure 7. On the top: Real part of the scattered pressure field $p(x, y) - e^{ikx}$ in a waveguide with a symmetric sudden expansion (localized at $kx = 5$ with amplitude 0.5 on both sides) at the frequency $kh = 2$. On the bottom: Real part of the plane mode $p_0(x) - e^{ikx}$, deduced from the reference calculation (black line), in the WA (doted red line) and in the IWA, including the contribution of the boundary mode (plain red line from Eq. (4.11)), with $D = 0.2964$.

5. Conclusion

In this paper, we have revisited an efficient method developed earlier which consists in adding an extra non physical mode to the usual modal expansion to get a better convergence of the modal series. By performing a change of unknowns we are able to partially decouple the modal components, which improves the boundary mode method and leads to at least two interesting consequences. 1) It allows to identify the nature of the boundary mode and its relation with the usual modes. This defines radiation conditions and thus facilitates the use of efficient numerical methods like the admittance matrix method. The numerical tests have shown that our method is very efficient in reducing the number of degrees of freedom: adding to the boundary modes, it is sufficient to take only the propagating modes to get very good results. Works are in progress to investigate in details the strength of such approach in multimodal numerical schemes (Maurel *et al.* 2013). 2) In the low frequency regime, that is the main goal of the present paper, the boundary mode is used to derive new approximate equations improving the Webster equation. Extensions to 3D axisymmetric waveguides and to bent waveguides with varying cross section are under progress.

Acknowledgments – We acknowledge the support from the Agence Nationale de la Recherche, ANR-10-INTB-0914 ProComedia. The authors thank Christophe Hazard and Eric Lunéville for useful discussions.

Appendix A. Projection of the wave equation

The derivation of the wave equation (2.9) (with $h_1'=0$) is here recovered, and compared to classical projection. For the classical projection, we refer to Pagneux's derivation (Pagneux *et al.* 1996), who avoided an error in Stevenson 1956 that is commented below. The direct projection of the wave equation onto the eigenfunctions ψ_n is classic Stevenson 1951, Pagneux *et al.* 1996, with

$$p(x,y)=\sum_n p_n(x)\psi_n(y;x),\;\; \partial_x p(x,y)=\sum_n r_n(x)\psi_n(y;x),$$

and $\psi_0(y;x)=1$, $\psi_n(y;x)=\sqrt{2}\cos n\pi y/h(x)$, satisfying $hr_n=(hp_n)'-h'p_m(f_{mn}-a_{mn})$, with $f_{mn}\equiv\psi_m(h;x)\psi_n(h;x)$. The projection of ∂_y^2 is, using the boundary condition at $y=h$:

$$\int_0^h dy\,\psi_n\partial_y^2 p \;\;=\int_0^h dy\,p\,\partial_y^2\psi_n+h'\partial_x p(x,h)\psi_n(h;x)=-(n\pi/h)^2p_n+f_{mn}h'r_m.$$

The projection of ∂_x^2 starts with:

$$\frac{d}{dx}\int_0^h dy\,(p\psi_n)=\int_0^h dy\,(\partial_x p\,\psi_n+p\partial_x\psi_n)+h'p(x,h)\psi_n(h;x), \tag{A.1}$$

and the difference in the obtained representation comes from the treatment of this derivative.

In the following, we need the following relations:

- Starting from the definition of $a_{mn} = \int_0^{h(x)} (y\partial_y \psi_n(y;x))\psi_m(y;x)$, we deduce $y\partial_y\psi_n(y;x) = a_{ln}\psi_l(y;x)$. Then, it follows $a_{ln} f_{lm} = 0$ for any (n,m), where we used $\partial_y\psi(h;x) = 0$.

- We also have $d_{mn} \equiv \int_0^{h(x)} dy\, (y\partial_y\psi_n(y;x))(y\partial_y\psi_m(y;x)) = a_{ln}\int_0^{h(x)} dy\, \psi_l(y;x)\, (y\partial_y\psi_m(y;x)) = a_{ln}a_{lm}$.

- Integrating by part $\int_0^{h(x)} dy\, y\psi_n(y;x)\psi_m(y;x)$, we have $a_{mn} + a_{nm} = -\delta_{mn} + f_{mn}$.

Classical derivation by direct projection of the wave equation

In Pagneux *et al.* 1996, the second derivative is, from Eq.(A.1):

$$\frac{d^2}{dx^2}\int_0^h dy(p\psi_n) = \int_0^h dy\, \partial_x^2\, p\psi_n + \left[(a_{nm}+\delta_{nm})h'' - d_{mn}\frac{h'^2}{h}\right] p_m \\ -(2a_{mn} - f_{mn}(1+h'^2))h' r_m,$$

using the boundary condition for p at $y = h(x)$ to get $\frac{d}{dx}p(x,h) = (1+h'^2)\partial_x p(x,h)$ and our $\psi_n(h;x)$ is independent of x. Note that it is this derivation that is done in Stevenson by deriving term by term, namely Stevenson considers $\frac{d}{dx}p(x,h) = \sum_m p'_m(x)\psi_m(h;x)$. Using rather $\left[-2a_{mn} + f_{mn}(1+h'^2)\right] hr_m = \left[-2a_{mn} + f_{mn}(1+h'^2)\right] p'_m + 2(h'/h)d_{mn}p_m$, we get the system of coupled differential equations on the p_n:

$$p''_n + k_n^2 p_n = p_m\left[\frac{h''}{h}a_{nm} + \frac{h'^2}{h^2}d_{nm}\right] + p'_m\frac{h'}{h}\left[2a_{nm} - f_{mn}(1+h'^2)\right]. \qquad \text{(A.2)}$$

Alternative projection

Alternatively, we can consider the derivation of Eq.(A.1):

$$\frac{d^2}{dx^2}\int_0^h dy\,(p\psi_n) = \; = \int_0^h dy\, \partial_x^2 p\, \psi_n (a_{mn}+\delta_{mn})h' r_m + \left[(a_{nm}+\delta_{mn})h' p_m\right]'.$$

Using $a_{mn}hr_m = a_{mn}hp'_m - d_{mn}h'p_m$, we get:

$$p''_n + k_n^2(x)p_n = d_{mn}\frac{h'^2}{h^2}p_m - (a_{mn}+\delta_{mn})\frac{h'}{h}p'_m + \frac{1}{h}\left[a_{nm}h'p_m\right]',$$

that corresponds to our Eq. (2.4).

Appendix B. Expressions of (α, β), (a, b, c, d)

For $m \le N$ and $n \le N$, we have obviously $\beta_n = 1$ and $\gamma_n = (n\pi)^2$. Then:

$$a_{mn} = \begin{cases} 0, \\ \sqrt{2}(-1)^n, \\ 1/2, \\ \dfrac{2(-1)^{n+m}n^2}{n^2-m^2}, \end{cases} \qquad b_{mn} = \begin{cases} 0, & \text{if } n = 0, \\ \sqrt{2}((-1)^n - 1); & \text{if } m = 0, n \neq 0, \\ 0 & \text{if } n = m, \\ ((-1)^{n+m} - 1)\dfrac{2n^2}{n^2-m^2} & \text{otherwise,} \end{cases}$$

$$c_{mn} = \begin{cases} 0, \\ \dfrac{n^2\pi^2}{2}, \\ ((-1)^{n+m} - 1)\dfrac{4n^2m^2}{(m^2-n^2)^2}, \end{cases} \qquad d_{mn} = \begin{cases} 0, & \text{if } m \text{ or } n = 0, \\ \dfrac{n^2\pi^2}{3} - \dfrac{1}{2}, & \text{if } m = n \neq 0, \\ 8\dfrac{(-1)^{n+m}n^2m^2}{(n^2-m^2)^2}, & \text{otherwise.} \end{cases}$$

for one degree of freddom

We have

$$\alpha_m \equiv (\varphi_m, \chi) = \begin{cases} \dfrac{2\sqrt{2}}{\pi}, & \text{if } m = 0, \\ \dfrac{(-1)^{m+1}}{\pi(m^2 - 1/4)}, & \text{if } m \neq 0. \end{cases} \tag{A.3}$$

The coefficients β_{N+1} and γ_{N+1} are given using $(\chi, \chi) = 1$, $(\chi', \chi') = (\pi/2)^2$ and $(\varphi_n, \varphi_m) = \delta_{mn}$, $(\varphi'_n, \varphi'_m) = (n\pi)^2\delta_{mn}$ for $(m, n) \le N$

$$\begin{aligned} \beta_{N+1} &= (\varphi_{N+1}, \varphi_{N+1}) = 1 - \sum_{n=0}^{N} \alpha_n^2, \\ \gamma_{N+1} &\equiv (\varphi'_{N+1}, \varphi'_{N+1}) = (\pi/2)^2 - \sum_{n=0}^{N} (n\pi\alpha_n)^2. \end{aligned} \tag{A.4}$$

We also need the coefficients a, b, c, d for the boundary mode (although not necessary for one varying wall, we define also the coefficients b and c, see following section).

$$\begin{aligned} a_{m,N+1} &= a_m - \sum_{n=0}^{N} \alpha_n a_{mn}, & a_{N+1,m} &= \hat{a}_m - \sum_{n=0}^{N} \alpha_n a_{nm}, \\ b_{m,N+1} &= b_m - \sum_{n=0}^{N} \alpha_n b_{mn}, & b_{N+1,m} &= \hat{b}_m - \sum_{n=0}^{N} \alpha_n b_{nm}, \\ c_{m,N+1} &= c_m - \sum_{n=0}^{N} \alpha_n c_{mn}, & d_{m,N+1} &= d_m - \sum_{n=0}^{N} \alpha_n d_{mn}, \end{aligned} \tag{A.5}$$

where $a_m \equiv (Y\varphi_m, \chi')$, $\hat{a}_m \equiv (Y\chi, \varphi'_m)$, $b_m \equiv (\varphi_m, \chi')$, $\hat{b}_m \equiv (\chi, \varphi'_m)$, $c_m \equiv (Y^2\chi', \varphi'_m)$ and $d_m \equiv (Y^2\chi', \varphi'_m)$.

$$a_m = \begin{cases} -\dfrac{2\sqrt{2}}{\pi}, & \text{if } m = 0, \\ \dfrac{-(-1)^m}{\pi}\dfrac{m^2+1/4}{(m^2-1/4)^2}, & \text{if } m \neq 0, \end{cases}, \quad \hat{a}_m = \frac{(-1)^m}{\pi}\frac{2m^2}{(m^2-1/4)^2},$$

$$b_m = \begin{cases} -\sqrt{2}, & \text{if } m = 0, \\ \dfrac{1}{2(m^2-1/4)}, & \text{if } m \neq 0, \end{cases}, \qquad \hat{b}_m = \begin{cases} 0, & \text{if } m = 0, \\ \dfrac{-2m^2}{m^2-1/4}, & \text{if } m \neq 0, \end{cases}, \tag{A.6}$$

$$c_m = \frac{m}{2}\left[\frac{1}{(m+1/2)^2} - \frac{1}{(m-1/2)^2} - \frac{2\pi m(-1)^m}{m^2-1/4}\right], \tag{A.7}$$

$$d_m = (-1)^m \frac{m^2}{m^2-1/4}\left[\frac{2}{\pi}\frac{m^2+3/4}{(m^2-1/4)^2} - 1\right],$$

The asymptotic forms of $a_{m,N+1}$, $a_{N+1,m}$ and $d_{m,N+1}$ can be checked numerically, but since they are rests of series, they can be evaluated explicitly. For instance, we have $d_{m,N+1} = \sum_{n=N+1}^{\infty} \alpha_n d_{mn}$

$$d_{m,N+1} = -\frac{2(-1)^m m^2}{\pi(m^2-1/4)^2} \sum_{n=N+1}^{\infty} \left[\frac{m^2-1/4}{(n-m)^2} + \frac{m^2-1/4}{(n+m)^2} + \frac{1}{(n^2-1/4)} - \frac{2(m^2+1/4)}{(n^2-m^2)}\right]. \tag{A.8}$$

We can now evaluate $\sum_{n=N+1}^{\infty} 1/(n^2-a^2) \sim \int_{N+1}^{\infty} dx/(x^2-a^2) \simeq 1/N + a^2/(3N^3)$, and $\sum_{n=N+1}^{\infty} 1/(n+a)^2 \sim \int_{N+1}^{\infty} dx/(x+a)^2 \simeq 1/(N+a) \simeq 1/N - a/N^2 + a^2/N^3$, for large N. It follows that $d_{m,N+1} \sim 8(-1)^m m^2/(3\pi N^3)$.

Finally, we also need $(Y\chi, \chi') = -1/2$, $(\chi, \chi') = -1$, $(Y\chi', \chi') = \pi^2/8 + 1/2$ and $(Y^2\chi', \chi') = \pi^2/12 + 1/2$ to get

$$\begin{aligned} a_{N+1,N+1} &= (Y\chi, \chi') - \sum_{n=0}^{N} \alpha_n (a_{n,N+1} + a_{N+1,n}) - \sum_{n,m=0}^{N} \alpha_n \alpha_m a_{nm}, \\ b_{N+1,N+1} &= (\chi, \chi') - \sum_{n=0}^{N} \alpha_n (b_{n,N+1} + b_{N+1,n}) - \sum_{n,m=0}^{N} \alpha_n \alpha_m b_{nm}, \\ c_{N+1,N+1} &= (Y\chi', \chi') - 2\sum_{n=0}^{N} \alpha_n c_{n,N+1} - \sum_{n,m=0}^{N} \alpha_n \alpha_m c_{nm}, \\ d_{N+1,N+1} &= (Y^2\chi', \chi') - 2\sum_{n=0}^{N} \alpha_n d_{n,N+1} - \sum_{n,m=0}^{N} \alpha_n \alpha_m d_{nm}, \end{aligned} \tag{A.9}$$

It is more convenient to express the coefficients as a function of the formers. We have

$$\alpha_m^{(1)} = \frac{\xi_1}{\sqrt{2}}(1+(-1)^m)\alpha_m, \quad \alpha_m^{(2)} = \frac{\xi_2}{\sqrt{2}}(1-(-1)^m)\alpha_m, \tag{A.10}$$

with α_m given in Eq. (A.3). We also have

$$\begin{aligned} \beta_{N+1} &= 1 - \sum_{n=0}^{N} {\alpha_n^{(1)}}^2, & \beta_{N+2} &= 1 - \sum_{n=0}^{N} {\alpha_n^{(2)}}^2, \\ \gamma_{N+1} &= (\pi/2)^2 - \pi\xi_1^2 - \sum_{n=0}^{N}(n\pi\alpha_m^{(1)})^2, & \gamma_{N+2} &= (\pi/2)^2 + \pi\xi_2^2 - \sum_{n=0}^{N}(n\pi\alpha_m^{(2)})^2. \end{aligned} \tag{A.11}$$

Then,we define, for $j = 1, 2$

$$\begin{aligned} a_{m,N+j} &= a_m^{(j)} - \sum_{n=0}^{N} \alpha_n^{(j)} a_{mn}, & a_{N+j,m} &= \hat{a}_m^{(j)} - \sum_{n=0}^{N} \alpha_n^{(j)} a_{nm}, \\ b_{m,N+j} &= b_m^{(j)} - \sum_{n=0}^{N} \alpha_n^{(j)} b_{mn}, & b_{N+j,m} &= \hat{b}_m^{(j)} - \sum_{n=0}^{N} \alpha_n^{(j)} b_{nm}, \\ c_{m,N+j} &= c_m^{(j)} - \sum_{n=0}^{N} \alpha_n^{(j)} c_{mn}, & d_{m,N+j} &= d_m^{(j)} - \sum_{n=0}^{N} \alpha_n^{(j)} d_{nm}, \end{aligned} \tag{A.12}$$

with $c_{m,N+j}$ and $d_{m,N+j}$ being the symmetrical forms. The coefficients $a^{(j)}, \hat{a}^{(j)}, b^{(j)}, \hat{b}^{(j)}, c^{(j)}, d^{(j)}$ can be expressed as a function of the coefficients $a, \hat{a}, b, \hat{b}, c, d$ in Eq. (A.7).

$$a_m^{(1)} = \begin{cases} \xi_1(1-4/\pi), \\ \dfrac{\xi_1}{\sqrt{2}}\left[a_m + \dfrac{c_m + \hat{b}_m}{2\pi m^2}\right], \end{cases} \quad a_m^{(2)} = \begin{cases} -\xi_2, & m = 0, \\ \dfrac{\xi_2}{\sqrt{2}}\left[a_m - \dfrac{c_m + \hat{b}_m}{2\pi m^2}\right], & m > 0 \end{cases}$$

$$\hat{a}_n^{(1)} = \frac{\xi_1}{\sqrt{2}}\left[\hat{a}_n - \frac{2}{\pi}c_n\right], \qquad \hat{a}_n^{(2)} = \frac{\xi_2}{\sqrt{2}}\left[\hat{a}_n + \frac{2}{\pi}c_n\right],$$

$$b_n^{(1)} = \frac{\xi_1}{\sqrt{2}}(b_n + \frac{\pi}{2}\alpha_n), \qquad b_n^{(2)} = \frac{\xi_2}{\sqrt{2}}(b_n - \frac{\pi}{2}\alpha_n),$$

$$\hat{b}_n^{(1)} = \frac{\xi_1}{\sqrt{2}}(\hat{b}_n - 2\pi n^2\alpha_n), \qquad \hat{b}_n^{(2)} = \frac{\xi_2}{\sqrt{2}}(\hat{b}_n + 2\pi n^2\alpha_n),$$

$$c_n^{(1)} = \frac{\xi_1}{\sqrt{2}}(\hat{c}_n + \frac{\pi}{2}\hat{a}_n), \qquad c_n^{(2)} = \frac{\xi_2}{\sqrt{2}}(\hat{c}_n - \frac{\pi}{2}\hat{a}_n),$$

$$d_n^{(1)} = -\frac{\xi_1}{2\sqrt{2}(n^2-1/4)}\left[4n^2 a_n - \hat{a}_n + \frac{2}{\pi}(2c_n + \hat{b}_n) - 2n^2(-1)^n\right],$$
$$d_n^{(2)} = -\frac{\xi_2}{2\sqrt{2}(n^2-1/4)}\left[4n^2 a_n - \hat{a}_n - \frac{2}{\pi}(2c_n + \hat{b}_n) - 2n^2(-1)^n\right]. \qquad \text{(A.13)}$$

The coefficients for $n = N + j$ or $m = N + j$, $j = 1, 2$ are of the form (we give the example for $a_{N+1,N+2}$)

$$a_{N+1,N+2} = (Y\chi_1, \chi_2') - \sum_{n=0}^{N} \alpha_n^{(1)} a_{n,N+2} - \sum_{m=0}^{N} \alpha_m^{(2)} a_{N+1,m} - \sum_{n,m=0}^{N} \alpha_n^{(1)}\alpha_m^{(2)} a_{nm}, \qquad \text{(A.14)}$$

and this can be done for all coefficients b, c, d. It appears that we need the following integrals $(Y\chi_1, \chi_1') = -\xi_1^2/\pi$, $(Y\chi_2, \chi_2') = \xi_2^2/\pi$, $(Y\chi_1, \chi_2') = -\xi_1\xi_2/(2+\pi)$, $(Y\chi_2, \chi_1') = -\xi_1\xi_2/(2-\pi)$ $(\chi_1, \chi_1') = 0$, $(\chi_2, \chi_2') = 0$, $(\chi_1, \chi_2') = -\pi\xi_2/(2\xi_1)$ $(\chi_2, \chi_1') = \pi\xi_1/(2\xi_2)$ $(Y\chi_1', \chi_1') = \xi_1^2\pi^2/(8(1-2/\pi))$, $(Y\chi_2', \chi_2') = \xi_2^2\pi^2/(8(1+2/\pi))$, $(Y\chi_1', \chi_2') = \xi_1\xi_2/2$, $(Y^2\chi_1', \chi_1') = -\xi_1^2\pi^2\left[1/12 + 1/\pi^3 - 1/4\pi\right]$, $(Y^2\chi_2', \chi_2') = -\xi_2^2\pi^2\left[1/12 - 1/\pi^3 1/4\pi\right]$, $(Y^2\chi_1', \chi_2') = -\xi_1\xi_2/2$.

Appendix C. On the Webster equation

The plane wave approximation, where $p(x, y) \simeq p_0(x)$ is known to produce the Webster equation:

$$p_0'' + k^2 p_0 = -\frac{h'}{h} p_0.$$

In the low frequency regime ($\epsilon \equiv kh \ll 1$) we have $p_0 = O(1)$ (incident mode). For $n \neq 0$, from Eq. (2.4), using $k_n^2 \simeq (n\pi/h)^2$ we deduce that $p_n = O(h'^2, \epsilon h')$. For $n = 0$, with $a_{00} = d_{00} = 0$, Eq. (2.4) leads to:

$$p_0'' + k^2 p_0 + \frac{h'}{h} p_0' = k^2 O\left(\frac{h'^3}{\epsilon^2}, \frac{h'^2}{\epsilon}\right),$$

from which the Webster equation can be deduced if $h'^2 \ll \epsilon \ll 1$. A particular case satisfying such condition is used in [**?**] with $h' = \epsilon$. Eq. (A.2) leads to a slightly different equation:

$$p_0'' + k^2 p_0 + \frac{h'}{h}(1 + h'^2) p_0' = k^2 O\left(\frac{h'^3}{\epsilon^2}, \frac{h'^2}{\epsilon}\right).$$

As it was already mentioned in Pagneux *et al.* 1996, Eq. (A.2) seems to have an extra term in h'^3. However the extra term:

$$\frac{h'^3}{h} p_0 = k^2 O\left(\frac{h'^3}{\epsilon}\right) \ll O\left(\frac{h'^3}{\epsilon^2}\right),$$

has to be neglected, leading to the usual Webster equation.

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