CAN TRAPPED MODES OCCUR IN OPEN WAVEGUIDES ?

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Wave propagation in complex media
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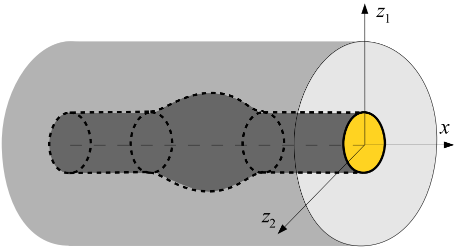
Introduction

CONTEXT:

time-harmonic waves in locally perturbed uniform open waveguides (for instance, a defect in an optical fiber, or in an immersed pipe ...).



ISSUE :

Are there trapped modes, i.e., localized oscillations of the system which do not radiate towards infinity?

Our 3-dimensional acoustic waveguide

Defined by a wavenumber function

$$k = k(x,z) \quad \text{where} \quad \begin{cases} x = \text{longitudinal direction}, \\ z := (z_1, z_2) = \text{transverse directions}, \end{cases}$$

such that

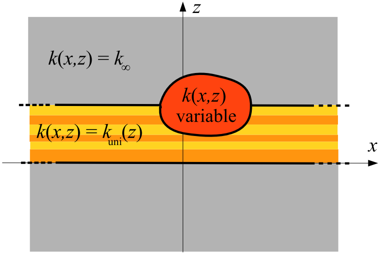
$$0 < \inf_{(x,z)\in\mathbb{R}^3} k(x,z) \quad \text{and} \quad \sup_{(x,z)\in\mathbb{R}^3} k(x,z) < \infty,$$

and k is a localized perturbation of a uniform waveguide:

$k - k_{\text{uni}}$ is compactly supported,

where $k_{\text{uni}} = k_{\text{uni}}(z)$ and

$$k_{\text{uni}}(z) = k_\infty > 0 \quad \text{if } |z| > d > 0.$$



Main result

Theorem (absence of trapped modes)

With the above assumptions on $k = k(x, z)$, the only solution $u \in H^2(\mathbb{R}^3)$ to the Helmholtz equation

$$-\Delta u - k^2\, u = 0 \quad \text{in } \mathbb{R}^3,$$

is $u \equiv 0$.

Basic ideas for the proof:

- Modal decomposition of u resulting from a generalized Fourier transform in the transverse direction (instead of a usual Fourier transform in the longitudinal direction).
- Argument of analyticity with respect to the generalized Fourier variable.

Related works

Rough media

- Chandler-Wilde and Zhang (1998)
- Chandler-Wilde and Monk (2005)
- Lechleiter and Ritterbusch (2010)
- . . .

} No guided wave

Perturbed stratified media

- Xu (1992)
- Ciraolo and Magnanini (2008)

} Incomplete proofs !

- Weder (1991)

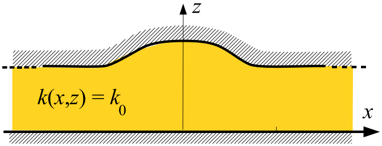
} Analyticity argument

- Bonnet-Ben Dhia, Chorfi, Dakia, H. (2009)
- Bonnet-Ben Dhia, Goursaud, H. (2011)

} 2D step-index

Outline

Closed waveguides



On each side of the bulge, any acoustic field can be decomposed as

$$u(x,z) = \sum \widehat{u}(x,\lambda_n)\ \Phi_{\lambda_n}(z) = \begin{cases} \text{finite sum of propagative modes} \\ + \\ \text{infinite sum of evanescent modes} \end{cases}$$

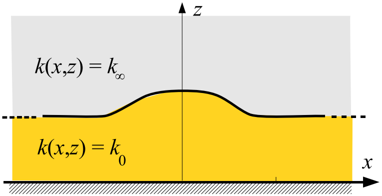
With no excitation :

Energy conservation $\Longrightarrow$ all propagative components vanish

But the evanescent components may not vanish!

Trapped modes may occur in closed waveguides (see, e.g., Linton and McIver (2007))

Open waveguides



On each side of the bulge, any acoustic field can be decomposed as

$$u(x,z) = \sum_{\text{finite}} \widehat{u}(x,\lambda_n)\ \Phi_{\lambda_n}(z) + \int \widehat{u}(x,\lambda)\ \Phi_{\lambda}(z)\ \mathrm{d}\lambda$$

$$= \begin{cases} \text{finite sum of guided modes (propagative along } x \\ \quad \text{and evanescent along } z) \\ \qquad\qquad + \\ \text{continuous superposition of radiation modes (either} \\ \text{propagative or evanescent along } x \text{ and propagative along } z) \end{cases}$$

Again, with no excitation :

Energy conservation $\Longrightarrow$ all x-propagative components vanish

Open waveguides (continued)

$$\Longrightarrow \quad u(x,z) = \int \widehat{u}(x,\lambda)\ \Phi_\lambda(z)\ \mathrm{d}\lambda \quad \text{(only } x\text{-evanescent modes)}$$

A key property: analyticity

For a localized bulge, $\widehat{u}(x,\lambda)$ extends to an analytic function for complex values of λ (for fixed x).

Hence, as $\widehat{u}(x,\lambda)$ vanishes for a continuous set of λ (x-propagative modes), it must vanish everywhere $\Longrightarrow$ $u = 0$.

Conclusion

Trapped modes cannot occur in open waveguides

Remark : The idea does not apply for closed waveguides because the set of λ_n is discrete.

Modes of a uniform waveguide

Separation of variables: $u(x,z) = \Phi(z)\, e^{px}$ for $p \in \mathbb{C}$ solution to

$$-\Delta_{x,z} u - k_{\rm uni}^2\, u = 0 \quad \text{in } \mathbb{R}^3,$$

$$\Longrightarrow \text{Eigenvalue problem} \quad \begin{cases} \text{Find } \lambda = p^2 \in \mathbb{C} \text{ and } \Phi \text{ bounded such that} \\ -\Delta_z \Phi - k_{\rm uni}^2\, \Phi = \lambda\, \Phi \text{ in } \mathbb{R}^2. \end{cases}$$

Assuming $k_\infty < k_{\sup} := \sup\limits_{z \in \mathbb{R}^2} k_{\rm uni}(z)$, there are two kinds of solutions:

- Finite set of isolated $\lambda \in (-k_{\sup}^2, -k_\infty^2)$ associated with evanescent Φ (as $|z| \to +\infty$).

  $\Longrightarrow$ Guided modes $\Phi(z)\, e^{\pm\sqrt{\lambda} x}$.

- Continuous set $\lambda \in [-k_\infty^2, +\infty)$ associated with oscillating Φ (as $|z| \to +\infty$)

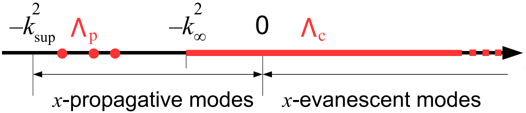
  $$\Longrightarrow \text{Radiation modes} \begin{cases} \text{propagative as } x \to \pm\infty \text{ if } \lambda < 0, \\ \text{exponentially } \nearrow \text{ or } \searrow \text{ if } \lambda > 0. \end{cases}$$

Spectral interpretation

The unbounded operator A defined in $L^2(\mathbb{R}^2)$ by

$$A\varphi := -\Delta_z \varphi - k_{\text{uni}}^2\, \varphi \quad \forall \varphi \in \mathrm{D}(A) := H^2(\mathbb{R}^2)$$

is selfadjoint. Its spectrum Λ is composed of two parts:



- A finite point spectrum $\Lambda_\text{p} = \{\text{eigenvalues}\} \subset (-k_{\text{sup}}^2, -k_\infty^2)$.
  $\Longrightarrow$ Associated $\Phi \in L^2(\mathbb{R}^2)$: eigenfunctions.

- A continuous spectrum $\Lambda_\text{c} = [-k_\infty^2, +\infty)$.
  $\Longrightarrow$ Associated $\Phi \notin L^2(\mathbb{R}^2)$: generalized eigenfunctions.

Towards a generalized spectral basis ?

Can we find a family of eigenfunctions and generalized eigenfunctions such that

- any $\varphi \in L^2(\mathbb{R}^2)$ can be represented by a discrete + continuous superposition, and
- A becomes diagonal in this "basis"?

Towards a generalized spectral basis ?

Can we find a family of eigenfunctions and generalized eigenfunctions such that

- any $\varphi \in L^2(\mathbb{R}^2)$ can be represented by a discrete + continuous superposition, and
- A becomes diagonal in this "basis"?

YES !

- easy for the eigenfunctions, but ...
- more involved for the generalized eigenfunctions ($\Longrightarrow$ scattering theory).

A family of eigenfunctions (guided modes)

For $\lambda \in \Lambda_{\mathrm{p}}$, choose an orthonormal basis $\{\Phi_{\lambda,\kappa};\ \kappa = 1,\ldots,m_\lambda\}$ of the associated eigenspace (m_λ = multiplicity of the eigenvalue λ).

A family of generalized eigenfunctions (radiation modes)

For $\lambda \in \Lambda_{\mathrm{c}} = [-k_\infty^2, +\infty)$ and $\kappa \in S^1$ (= unit circle), we define

$$\Phi_{\lambda,\kappa} := \Phi^{\infty}_{\lambda,\kappa} + \Phi^{\mathrm{sc}}_{\lambda,\kappa} \quad \text{where}$$

- $\Phi^{\infty}_{\lambda,\kappa}(z) := (8\pi^2)^{-1/2} \exp\left(\mathrm{i}\sqrt{\lambda + k_\infty^2}\,\kappa \cdot z\right)$ is an incident plane wave of direction κ, solution to

$$-\Delta_z \Phi^{\infty}_{\lambda,\kappa} - \left(k_\infty^2 + \lambda\right)\Phi^{\infty}_{\lambda,\kappa} = 0 \text{ in } \mathbb{R}^2,$$

- $\Phi^{\mathrm{sc}}_{\lambda,\kappa}(z)$ is the associated outgoing scattered wave, solution to

$$\begin{cases} -\Delta_z \Phi^{\mathrm{sc}}_{\lambda,\kappa} - \left(k_{\mathrm{uni}}^2 + \lambda\right)\Phi^{\mathrm{sc}}_{\lambda,\kappa} = \left(k_{\mathrm{uni}}^2 - k_\infty^2\right)\Phi^{\infty}_{\lambda,\kappa} \text{ in } \mathbb{R}^2, \\ \text{radiation condition as } |z| \to \infty. \end{cases}$$

The generalized Fourier transform

The operator of decomposition on the family $\{\Phi_{\lambda,\kappa}\}$:

$$(\mathcal{F}\varphi)(\lambda,\kappa) := \int_{\mathbb{R}^2} \varphi(z)\,\Phi_{\lambda,\kappa}(z)\,\mathrm{d}z \quad \forall\lambda\in\Lambda,\ \forall\kappa\in\begin{cases} 1,\ldots,m_\lambda & \text{if } \lambda\in\Lambda_\mathrm{p} \\ S^1 & \text{if } \lambda\in\Lambda_\mathrm{c}\end{cases}$$

defines (by density) a unitary transformation from $L^2(\mathbb{R}^2)$ to the spectral space

$$\widehat{\mathcal{H}} := \widehat{\mathcal{H}}_\mathrm{p} \oplus \widehat{\mathcal{H}}_\mathrm{c} \quad \text{where} \quad \widehat{\mathcal{H}}_\mathrm{p} := \oplus_{\lambda\in\Lambda_\mathrm{p}} \mathbb{C}^{m_\lambda} \text{ and } \widehat{\mathcal{H}}_\mathrm{c} := L^2(\Lambda_\mathrm{c}\times S^1).$$

$\mathcal{F}^{-1} = \mathcal{F}^*$ is the operator of recomposition on the family $\{\Phi_{\lambda,\kappa}\}$:

$$\mathcal{F}^{-1}\widehat{\varphi} = \sum_{\lambda\in\Lambda_\mathrm{p}} \sum_{\kappa=1}^{m_\lambda} \widehat{\varphi}(\lambda,\kappa)\,\Phi_{\lambda,\kappa} + \int_{\Lambda_\mathrm{c}\times S^1} \widehat{\varphi}(\lambda,\kappa)\,\Phi_{\lambda,\kappa}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_\kappa .$$

It diagonalizes A in the sense that $A = \mathcal{F}^{-1}\lambda\,\mathcal{F}$.

A key property: analyticity

For all fixed $\kappa \in S^1$ and $z \in \mathbb{R}^2$, the function $\lambda \mapsto \Phi_{\lambda,\kappa}(z)$ extends to a meromorphic function of λ in the complex half plane $\operatorname{Re}\lambda > -k_\infty^2$.

Corollary

For all fixed $\kappa \in S^1$ and $\varphi \in L^2(\mathbb{R}^2)$ with compact support, the function $\lambda \mapsto \mathcal{F}\varphi(\lambda, \kappa)$ extends to a meromorphic function of λ in the complex half plane $\operatorname{Re}\lambda > -k_\infty^2$.

Getting rid of the defect!

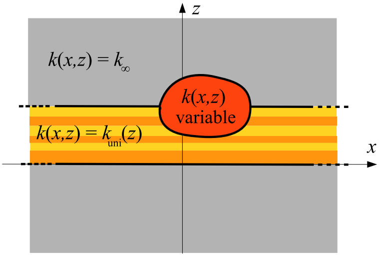
If $u \in H^2(\mathbb{R}^3)$ satisfies

$$\text{(H)} \quad -\Delta u - k^2\, u = 0 \quad \text{in } \mathbb{R}^3,$$

then

$$\text{(LS)} \quad -\Delta u - k_{\text{uni}}^2\, u = f(u) \quad \text{in } \mathbb{R}^3,$$

where $f(u) := (k^2 - k_{\text{uni}}^2)u$ is compactly supported.



Proof of the absence of trapped modes

1) Prove: (LS) $\Longrightarrow$ $u = 0$ outside the support of $f(u)$.
2) Conclude by the unique continuation principle for (H).

Main theorem

Let $f \in L^2(\mathbb{R}^3)$ compactly supported. If $u \in H^2(\mathbb{R}^3)$ satifies

$$-\Delta u - k_{\text{uni}}^2\, u = f \quad \text{in } \mathbb{R}^3,$$

then $u = 0$ outside the support of f.

The steps of the proof:

- use $\mathcal{F} \implies$ modal representation of u;
- finite energy $\implies$ the modal components of u associated with propagative modes must vanish (outside the support of f);
- analyticity of the modal components $\implies$ the other modal components (associated with evanescent modes) must also vanish.

$\implies u = 0$ outside the support of f.

Using $\mathcal{F}$

Let $f \in L^2(\mathbb{R}^3)$ compactly supported and $u \in H^2(\mathbb{R}^3)$ solution to

$$-\Delta u - k_{\text{uni}}^2\, u = f \quad \text{in } \mathbb{R}^3.$$

In other words,

$$-\frac{\partial^2 u}{\partial x^2} + Au = f \quad \text{in } \mathbb{R}.$$

Setting $\widehat{u}_{\lambda,\kappa}(x) := (\mathcal{F}u(x,\cdot))(\lambda,\kappa)$ and $\widehat{f}_{\lambda,\kappa}(x) := (\mathcal{F}f(x,\cdot))(\lambda,\kappa)$ (which makes sense since $u, f \in L^2(\mathbb{R}^3)$), we have

$$-\frac{\partial^2 \widehat{u}_{\lambda,\kappa}}{\partial x^2} + \lambda\, \widehat{u}_{\lambda,\kappa} = \widehat{f}_{\lambda,\kappa} \quad \text{in } \mathbb{R}, \quad \text{for a.e. } \lambda \text{ and } \kappa.$$

Using $\mathcal{F}$ (contd)

Any solution to $-\dfrac{\partial^2 \widehat{u}_{\lambda,\kappa}}{\partial x^2} + \lambda\, \widehat{u}_{\lambda,\kappa} = \widehat{f}_{\lambda,\kappa}$ reads as

$$\widehat{u}_{\lambda,\kappa} = \widehat{u}^{\text{gen}}_{\lambda,\kappa} + \widehat{u}^{\text{part}}_{\lambda,\kappa}$$

where

$$\widehat{u}^{\text{gen}}_{\lambda,\kappa}(x) = \widehat{\alpha}^{+}_{\lambda,\kappa}\, \mathrm{e}^{-\sqrt{\lambda}\, x} + \widehat{\alpha}^{-}_{\lambda,\kappa}\, \mathrm{e}^{+\sqrt{\lambda}\, x},$$

and

$$\widehat{u}^{\text{part}}_{\lambda,\kappa}(x) = \int_{\mathbb{R}} \gamma_{\lambda}(x - x')\, \widehat{f}_{\lambda,\kappa}(x')\, \mathrm{d}x',$$

where $\gamma_{\lambda}(x) := \dfrac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{2\sqrt{\lambda}}$ is a Green's function of $-\dfrac{\partial^2}{\partial x^2} + \lambda$ (choose $\sqrt{\lambda}$ such that $\sqrt{\lambda} \in \mathbb{R}^+$ if $\lambda \in \mathbb{R}^+$).

Using $\mathcal{F}$ (contd)

Outside the x-support of f,

$$\widehat{u}^{\text{part}}_{\lambda,\kappa}(x) = \widehat{\beta}^{\pm}_{\lambda,\kappa}\,\mathrm{e}^{-\sqrt{\lambda}\,|x|} \quad \text{as } x \to \pm\infty,$$

where

$$\widehat{\beta}^{\pm}_{\lambda,\kappa} := \int_{x\text{-supp}\,f} \frac{\mathrm{e}^{\pm\sqrt{\lambda}\,x'}}{2\sqrt{\lambda}}\,\widehat{f}_{\lambda,\kappa}(x')\,\mathrm{d}x'.$$

So

$$\widehat{u}_{\lambda,\kappa}(x) = \begin{cases} \widehat{\alpha}^{+}_{\lambda,\kappa}\,\mathrm{e}^{-\sqrt{\lambda}\,x} + \left(\widehat{\alpha}^{-}_{\lambda,\kappa} + \widehat{\beta}^{-}_{\lambda,\kappa}\right)\mathrm{e}^{+\sqrt{\lambda}\,x} & \text{as } x \to -\infty, \\ \left(\widehat{\alpha}^{+}_{\lambda,\kappa} + \widehat{\beta}^{+}_{\lambda,\kappa}\right)\mathrm{e}^{-\sqrt{\lambda}\,x} + \widehat{\alpha}^{-}_{\lambda,\kappa}\,\mathrm{e}^{+\sqrt{\lambda}\,x} & \text{as } x \to +\infty. \end{cases}$$

Solutions with finite energy

Recall that $\mathcal{F}$ is unitary, hence

$$u \in L^2(\mathbb{R}^3) \Longrightarrow \widehat{u}_{\lambda,\kappa} \in L^2(\mathbb{R}) \quad \text{for a.e. } \lambda \text{ and } \kappa.$$

Among the possible $\widehat{u}_{\lambda,\kappa} = \widehat{u}^{\text{gen}}_{\lambda,\kappa} + \widehat{u}^{\text{part}}_{\lambda,\kappa}$, which ones belong to $L^2(\mathbb{R})$?

- Propagative modes: $\lambda < 0$.
  As $x \to \pm\infty$, $\widehat{u}_{\lambda,\kappa}$ = linear combination of oscillating exp. functions
  $$\Longrightarrow \begin{cases} \widehat{\alpha}^{+}_{\lambda,\kappa} = \widehat{\alpha}^{-}_{\lambda,\kappa} + \widehat{\beta}^{-}_{\lambda,\kappa} = 0, \\ \widehat{\alpha}^{+}_{\lambda,\kappa} + \widehat{\beta}^{+}_{\lambda,\kappa} = \widehat{\alpha}^{-}_{\lambda,\kappa} = 0, \end{cases}$$
  $$\Longrightarrow \widehat{\alpha}^{\pm}_{\lambda,\kappa} = \widehat{\beta}^{\pm}_{\lambda,\kappa} = 0.$$

- Evanescent modes: $\lambda > 0$.
  As $x \to \pm\infty$, only decreasing exp. functions are allowed
  $$\Longrightarrow \widehat{\alpha}^{+}_{\lambda,\kappa} = \widehat{\alpha}^{-}_{\lambda,\kappa} = 0.$$

Solutions with finite energy (contd)

To sum up:

The only solutions with finite energy write as

$$\widehat{u}_{\lambda,\kappa}(x) = \widehat{u}^{\text{part}}_{\lambda,\kappa}(x) = \int_{\mathbb{R}} \gamma_\lambda(x - x')\, \widehat{f}_{\lambda,\kappa}(x')\, \mathrm{d}x'$$

with the condition

$$\widehat{u}_{\lambda,\kappa}(x) = 0 \quad \text{for } \lambda < 0,\ \kappa \in S^1 \text{ and } x \text{ outside the } x\text{-support of } f.$$

(i.e., the modal components of u associated with propagative modes vanish).

Analyticity of the modal components

$$\begin{aligned}\widehat{u}_{\lambda,\kappa}(x) &= \int_{\mathbb{R}} \frac{\mathrm{e}^{-\sqrt{\lambda}\,|x-x'|}}{2\sqrt{\lambda}}\, \widehat{f}_{\lambda,\kappa}(x')\,\mathrm{d}x' \\ &= \int_{\mathbb{R}} \frac{\mathrm{e}^{-\sqrt{\lambda}\,|x-x'|}}{2\sqrt{\lambda}} \int_{\mathbb{R}^2} f(x',z)\,\Phi_{\lambda,\kappa}(z)\,\mathrm{d}z\,\mathrm{d}x'\end{aligned}$$

Noticing that

- For all fixed $\kappa \in S^1$ and $z \in \mathbb{R}^2$, the function $\lambda \mapsto \Phi_{\lambda,\kappa}(z)$ extends to a meromorphic function of λ in the complex half plane $\mathrm{Re}\,\lambda > -k_\infty^2$,
- $\lambda \mapsto \sqrt{\lambda}$ is analytic outside the branch cut,
- f is compactly supported,

we deduce that

for all fixed $\kappa \in S^1$ and $x \in \mathbb{R}$, the function $\lambda \mapsto \widehat{u}_{\lambda,\kappa}(x)$ extends to a meromorphic function of λ in the complex half plane $\mathrm{Re}\,\lambda > -k_\infty^2$ outside the branch cut of $\sqrt{\lambda}$.

Analyticity of the modal components (contd)

We already know that the modal components of u associated with propagative modes vanish:

$$\widehat{u}_{\lambda,\kappa}(x) = 0 \quad \text{for } \lambda < 0, \ \kappa \in S^1 \text{ and } x \text{ outside the } x\text{-support of } f.$$

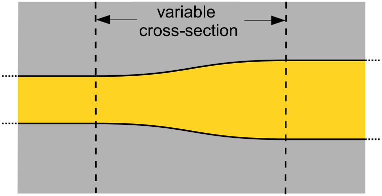
The analyticity of $\lambda \mapsto \widehat{u}_{\lambda,\kappa}(x)$ then shows that this holds for $\lambda \in \Lambda_c$, i.e., the modal components of u associated with evanescent modes also vanish.

Finally:

$u(x,z) = 0$ for all x outside the x-support of f and all $z \in \mathbb{R}^2$.

Conclusion

The same result seems to hold for the junction of two semi-infinite uniform open waveguides:



Theorem (absence of trapped modes)

The only solution $u \in H^2(\mathbb{R}^3)$ to the Helmholtz equation

$$-\Delta u - k^2 u = 0 \quad \text{in } \mathbb{R}^3,$$

is $u \equiv 0$.

Proof: sames ideas as for the defect, but... far more intricate!

Conclusion (contd)

The idea to remember:

Energy deals with propagative modes,
whereas analyticity takes care of evanescent modes.

Here, analyticity means that propagative and evanescent components of a radiating wave are connected in a subtle but strong way in an open waveguide (whereas they are independent in a closed waveguide).

Conclusion (contd)

What about scattering in open waveguides?

Case of 2D step-index waveguides:

- Bonnet-Ben Dhia, Chorfi, Dakia, H. (2009) = defect
- Bonnet-Ben Dhia, Goursaud, H. (2011) = junction

Use of $\mathcal{F} \Longrightarrow$ Modal radiation condition + well-posedness.

More general waveguides?
Main difficulty: extension of the generalized Fourier transform to slowly decreasing functions (not in $L^2(\mathbb{R}^2)$).

Thank you for your (trapped?) attention !