On the use of sampling methods to identify cracks in acoustic waveguides

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Abstract. We consider the identification of cracks in an acoustic 2D/3D waveguide with the help of sampling methods such as the Linear Sampling Method or the Factorization Method. A modal version of these sampling methods is used. Our paper emphasizes the fact that if one *a priori* knows the type of boundary condition which actually applies on the crack, then we shall adapt the formulation of our sampling method to such boundary condition in order to improve the efficiency of the method. The need for such adaptation is proved theoretically and illustrated numerically with the help of 2D examples. We also show by using our modal formulation that the Factorization Method is applicable in a waveguide with the same data as the Linear sampling Method.

1. Introduction

Inverse scattering consists in identifying some obstacle within a reference domain by measuring the scattered waves which result from the interaction between several known incident waves and this obstacle. The so-called "qualitative" or "sampling" methods introduced in [13] and [16] have considerably improved inverse scattering in acoustics, electromagnetism and elasticity, in particular in the frequency domain. These techniques have reached a high level of performance and generality, as can be seen in the recent monographs [11] and [17] that are devoted to the Linear Sampling Method and the Factorization Method, respectively. There are two special cases which introduce some additional complexity in the application of the sampling methods. The first one concerns the obstacles with empty interior, that is cracks. In such case, the justification of the sampling method is a bit more difficult than with impenetrable obstacles with non empty interior [18, 11], particularly when the crack is characterized by an impedance condition [3]. The second concerns domains which are bounded in one or two directions of space, that is waveguides. In such case, the identification of obstacles with qualitative methods is more challenging than in free space, because the scattered field contains an evanescent part that decays exponentially at long distance [10]. The case of an acoustic waveguide which is bounded in two directions is addressed in [23, 12, 10], while the case of an acoustic waveguide which is bounded in one direction is treated in [7, 1]. The case of an elastic waveguide which is bounded in two directions is analyzed in [8]. Another issue arising in the case of waveguides is the fact that strictly speaking the Factorization Method is applicable only by using incident waves that are "unphysical" since they are defined as the complex conjugate of a point source. Such point is raised in [17] (see paragraph 1.7) and discussed in detail in [1].

Our paper concerns the identification of cracks in acoustic waveguides, that is we address the two difficulties above at the same time. More precisely, we consider a modal formulation of sampling methods, which is specific to the waveguide geometry, that is the incident waves do not consist of point sources like in a classical near field formulation but consist of guided modes. Such modal formulation was first introduced in [10, 9] for the Linear Sampling Method in acoustics and extended to elasticity in [8]. The main advantage of the modal formulation is that it enables us to properly define a far field formulation, in other words a formulation which is based on measurements at long distance from the defects, which is important for non destructive testing applications. Our contribution in acoustics can be considered as a first step to address the more realistic and interesting problem of the crack detection in elastic waveguides. Indeed, the industrial applications in ultrasonic non destructive testing concern elastic structures, and most often the expected defect is a crack. A typical example of application, in rail transport, is the NDT of rails. In nuclear power plants, there are also a lot of metallic pipes that have to be inspected regularly. Note that the extension of the modal formulation from acoustics to elasticity requires the introduction of some special vector variables that mix the components of displacement and the components of the column of the stress tensor which is associated with the direction of propagation [2]. Since such developments in elasticity are quite technical, the identification of cracks in an elastic waveguide with the help of a modal formulation of the Linear Sampling Method will be explained in a future contribution.

The main objective of the present paper is to emphasize the fact that whenever the boundary condition on the crack is *a priori* known then the test function used in the formulation of the sampling method has to be properly chosen in order to optimize the quality of the reconstruction. This choice is illustrated for cracks that are known to be *a priori* of Dirichlet or Neumann type. In elasticity such choice would be crucial for applications. If for instance we consider non destructive testing for metallic materials, the defects that one tries to identify are traction free cracks. Hence in this particular case the boundary condition on the crack is known and such *a priori* information has to be taken into account to obtain good results for imaging.

A secondary objective of the present paper is to show that within the formalism of the modal formulation, we can apply the Factorization Method for a waveguide which is bounded in two directions by using the same data as for the Linear Sampling Method. This is in contrast with [12, 1] since in these two papers some "unphysical" incident waves were used in the

Factorization Method.
Our paper is organized as follows. In section 2 we introduce the forward and inverse problems. The Linear Sampling Method and the Factorization Method are then introduced in section 3, in particular in the modal form. We complete this section by some numerical experiments.

2. The forward and inverse problems

We consider a waveguide of domain $W = S \times \mathbb{R}$ in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. In 2D, we assume that $S = (-h, h)$, where $h > 0$, while in 3D, S is a bounded and open domain of $\mathbb{R}^2$, the boundary of which is smooth and denoted Γ. In the following, $x = (x_S, x_3)$ will denote a generic point of W, where $x_S \in S$ and $x_3 \in \mathbb{R}$.
Let us denote (θ_n, k_n^2), $n \in \mathbb{N}^*$, the eigenfunctions and eigenvalues of the Neumann eigenvalue problem for the negative Laplacian in S. The sequence $k_n^2 \in \mathbb{R}^+$ for $n \in \mathbb{N}^*$ is increasing, with $k_1 = 0$ and $k_n \to +\infty$ when $n \to +\infty$, and we can choose the θ_n such that they form an orthonormal basis of $L^2(S)$. It is straightforward to prove that the solutions of the problem

$$\left\{ \begin{array}{ccc} (\Delta + k^2)u = 0 & \text{in} & W \\ \partial_\nu u = 0 & \text{on} & \Gamma, \end{array} \right. \tag{1}$$

where ν is the outward unit normal on Γ, are the linear combinations of the so-called guided modes, which are the functions defined for $n \in \mathbb{N}^*$ by

$$g_n^\pm(x_S, x_3) = \theta_n(x_S) e^{\pm i\beta_n x_3}, \tag{2}$$

where β_n is defined by

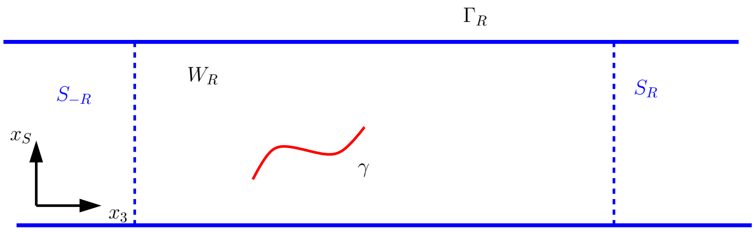
$$\beta_n = \sqrt{k^2 - k_n^2}, \quad \operatorname{Re}\beta_n, \operatorname{Im}\beta_n \geq 0. \tag{3}$$

In the following, we assume that

Assumption 2.1. *k is such that $\beta_n \neq 0$ for all $n > 0$.*

Thank's to such assumption, the guided modes are divided into n_p propagating modes, for which $\operatorname{Re}\beta_n > 0$, and an infinite number of evanescent modes, for which $\operatorname{Im}\beta_n > 0$. Moreover, the guided modes g_n^+ (respectively g_n^-) are either oscillating or decaying exponentially from the left to the right of the waveguide (respectively from the right to the left).

We are now in a position to introduce the forward problem for both the Dirichlet and Neumann crack problems. Following [11], let us denote by γ a portion of a smooth nonintersecting curve ($d = 2$) or surface ($d = 3$) that encloses a domain D in W, such that its boundary $\partial\gamma$ is smooth too ($d = 3$), with $\overline{\gamma} \in W$. We assume that γ is an open set with respect to the induced topology on ∂D. Such a manifold γ will be called a crack in the following. The normal vector ν on γ is defined as the outward normal vector to D.
We denote by $H^{\frac{1}{2}}(\gamma)$ the set of all restrictions to γ of functions in $H^{\frac{1}{2}}(\partial D)$, $\tilde{H}^{\frac{1}{2}}(\gamma)$ the



subspace of $H^{\frac{1}{2}}(\gamma)$ which consists of functions on γ such that their extension by 0 on ∂D belong to $H^{\frac{1}{2}}(\partial D)$. We denote by $H^{-\frac{1}{2}}(\gamma)$ and $\tilde{H}^{-\frac{1}{2}}(\gamma)$ the dual spaces of $\tilde{H}^{\frac{1}{2}}(\gamma)$ and $H^{\frac{1}{2}}(\gamma)$, respectively (see for example [11]). Note that the space $H^{-\frac{1}{2}}(\gamma)$ can be identified as the set of all restrictions to γ of distributions in $H^{-\frac{1}{2}}(\partial D)$, while $\tilde{H}^{-\frac{1}{2}}(\gamma)$ can be identified as the set of all distributions of $H^{-\frac{1}{2}}(\partial D)$ the support of which is contained in $\overline{\gamma}$.
Denoting now $S_s = S \times \{s\}$ any transverse section, we assume that $\overline{\gamma}$ lies between sections S_{-R} and S_R, with $R > 0$. Then W_R and Γ_R denote the portions of W and Γ which are limited by S_{-R} and S_R. The sequence of eigenfunctions $(\theta_n)_n$ and eigenvalues $(k_n^2)_n$ enables us to define a Dirichlet to Neumann linear and continuous operator $T_\pm$ acting on transverse sections $S_{\pm R}$, namely $T_\pm \; : H^{\frac{1}{2}}(S_{\pm R}) \longrightarrow \tilde{H}^{-\frac{1}{2}}(S_{\pm R})$, with for $h \in H^{\frac{1}{2}}(S_{\pm R})$

$$T_\pm h = \sum_{n>0} i\beta_n (h, \theta_n)_{S_{\pm R}} \theta_n,$$

where $(\cdot,\cdot)_{S_s}$ is the standard scalar product in $L^2(S_s)$. For $k > 0$, $f \in H^{\frac{1}{2}}(\gamma)$ and $g \in H^{-\frac{1}{2}}(\gamma)$, the forward Dirichlet/Neumann problem we consider in $W_R \setminus \overline{\gamma}$ is

$$\left\{ \begin{array}{ccc} (\Delta + k^2)u = 0 & \text{in} & W_R \setminus \overline{\gamma} \\ \partial_\nu u = 0 & \text{on} & \Gamma_R \\ u_\pm = f \quad \text{or} \quad \partial_\nu u_\pm = g & \text{on} & \gamma \\ \partial_\nu u = T_\pm u & \text{on} & S_{\pm R}, \end{array} \right. \tag{4}$$

where $u_\pm$ and $\partial_\nu u_\pm$ denote the trace of function u and the trace of its normal derivative on both side of the crack, where the sign $\pm$ is specified by the orientation of the normal ν on γ. The solution of problem (4) is the scattered field u^s associated with the incident field u^i with $f = -u^i|_\gamma$ or $g = -\partial_\nu u^i|_\gamma$. The last condition of system (4) is the radiation condition. We have the following theorem.

Theorem 2.2. *For $f \in H^{\frac{1}{2}}(\gamma)$ and $g \in H^{-\frac{1}{2}}(\gamma)$, the Dirichlet crack problem and the Neumann crack problem defined by (4) are well-posed in $H^1(W_R \setminus \overline{\gamma})$, except for at most a countable set of k.*

Proof. The proof is classical (see for example [6, 4]), so that we just give a sketch of it, in the case of the Neumann crack problem. The treatment of the Dirichlet case is very similar. It is easy to prove that an equivalent weak formulation to problem (4) is:

$$\text{Find} \quad u \in V := H^1(W_R \setminus \overline{\gamma}) \quad \text{such} \quad \text{that} \quad a(u,v) = l(v), \quad \forall v \in V, \tag{5}$$

where the continuous sesquilinear and antilinear forms a and l are defined by

$$a(u,v) := \int_{W_R\setminus\overline{\gamma}} (\nabla u\cdot\nabla\overline{v} - k^2 u\,\overline{v})\,dx - \int_{S_R} T_+ u\,\overline{v}\,ds - \int_{S_{-R}} T_- u\,\overline{v}\,ds,$$

$$l(v) := -\int_\gamma g(\overline{v_+} - \overline{v_-})\,ds,$$

where the integrals on $S_{\pm R}$ have the meaning of duality pairing between $\tilde{H}^{-\frac{1}{2}}(S_{\pm R})$ and $H^{\frac{1}{2}}(S_{\pm R})$, and the integral on γ has the meaning of duality pairing between $H^{-\frac{1}{2}}(\gamma)$ and $\tilde{H}^{\frac{1}{2}}(\gamma)$.
The sesquilinear form a may be written $a = b + c$, with

$$b(u,v) := \int_{W_R\setminus\overline{\gamma}} (\nabla u\cdot\nabla\overline{v} + u\,\overline{v})\,dx - \sum_{n>0} i\beta_n (u,\theta_n)_{S_{\pm R}}\,\overline{(v,\theta_n)_{S_{\pm R}}},$$

$$c(u,v) := -(1+k^2)\int_{W_R\setminus\overline{\gamma}} u\,\overline{v}\,dx.$$

The weak formulation (5) is of Fredholm type. Actually, by the Riesz theorem the form b defines a isomorphism on V since $\operatorname{Re} b(u,u) \geq ||u||^2_{H^1(V)}$, while the form c defines a compact operator on V since the mapping $H^1(W_R\setminus\overline{\gamma}) \to L^2(W_R\setminus\overline{\gamma})$ is compact. We conclude that uniqueness implies existence for the problem (4), or equivalently, for problem (5).
Hence, let us assume that $u \in V$ satisfies the weak formulation (5) with $g = 0$ and let us choose $v = u$. Taking the imaginary part of the obtained equation implies that $(u,\theta_n) = 0$ for $n = 1,\dots,n_p$. We hence obtain

$$a_H(k;u,v) = \lambda \int_{W_R\setminus\overline{\gamma}} u\,\overline{v}\,dx, \quad \forall v \in V, \tag{6}$$

where $\lambda = k^2$ and the hermitian sesquilinear form a_H is defined by

$$a_H(k;u,v) := \int_{W_R\setminus\overline{\gamma}} \nabla u\cdot\nabla\overline{v}\,dx + \sum_{n>n_p} \sqrt{k_n^2 - k^2}\,(u,\theta_n)_{S_{\pm R}}\,\overline{(v,\theta_n)_{S_{\pm R}}}.$$

By using again the compactness of the mapping $H^1(W_R\setminus\overline{\gamma}) \to L^2(W_R\setminus\overline{\gamma})$, well known results for that kind of variational eigenvalue problems (see for example the chapter 6 of [21]) imply that the eigenvalues λ satisfying (6) for $u \neq 0$ form a non-negative, increasing sequence $(\lambda_m)_{m>0}$ which tends to $+\infty$. Moreover such eigenvalues have the following $\min-\max$ characterization for all $m > 0$ (see for example [22]):

$$\lambda_m(k) = \min_{V_m\subset\mathcal{V}_m}\ \max_{v\in V_m,\,v\neq 0} \frac{a_H(k;v,v)}{\int_{W_R\setminus\overline{\gamma}} |v|^2\,dx},$$

where $\mathcal{V}_m$ denotes the set of all m-dimensional subspaces of V. For any integer $m > 0$, the function $k \mapsto \lambda_m(k)/k$ is non-increasing from the $\min-\max$ characterization, and continuous by the same arguments as in [6].

Hence the fixed-point equation $\lambda_m(k) = k^2 \Leftrightarrow \lambda_m(k)/k = k$ has exactly one solution k_m for each m. As a conclusion, uniqueness holds except for a set of values of k among the k_m. $\square$

Remark 2.3. *Note that some examples of non uniqueness exist, the non vanishing solutions of problem (4) for $f = 0$ or $g = 0$ being called trapped modes. For instance, such trapped modes are obtained for the Neumann case and horizontal crack in [5].*

Hence we introduce the following assumption, which is supposed to hold throughout this paper.

Assumption 2.4. *k is such that problem (4) is well-posed.*

We are now in a position to formulate the inverse problem we are interested in, with $\hat{S} := S_{-R} \cup S_R$.

The inverse problem (IP). *Given the measurements on $\hat{S}$ of the scattered fields $u_n^\pm$ associated with the incident fields $g_n^\pm$ for all $n > 0$, reconstruct the crack γ.*

One should note that in the inverse problem above, the incident waves do not consist of point sources as usual, they consist of guided modes. However, as shown in [10], the knowledge of the scattered fields associated with all point sources located on $\hat{S}$ is equivalent to the knowledge of the scattered fields associated with all guided modes. But the choice of guided modes as incident waves has the following advantage: in practice only the propagating modes $g_n^\pm$, which correspond to $n = 1, \cdots, n_p$, shall be used since the evanescent ones, which correspond to $n > n_p$, vanish exponentially at long distance. This will hereafter enable us to establish a far field formulation of sampling methods.

3. The sampling methods

3.1. Some preliminary results

In order to tackle the inverse problem above, we first consider some more classical incident fields $u^i = G(\cdot, y)$ or $u^i = \overline{G(\cdot, y)}$ for some $y \in \hat{S}$, where $G(\cdot, y)$ denotes the Green function of the waveguide W, which solves

$$\begin{cases} -(\Delta + k^2)G(\cdot, y) = \delta_y & \text{in} \quad W_R \\ \partial_\nu G(\cdot, y) = 0 & \text{on} \quad \Gamma_R \\ \partial_\nu G(\cdot, y) = T_\pm G(\cdot, y) & \text{on} \quad S_{\pm R}, \end{cases}$$

and is defined for all $x, y \in W$, by

$$G(x, y) = -\sum_{n>0} \frac{e^{i\beta_n |x_3 - y_3|}}{2i\beta_n} \theta_n(x_S)\theta_n(y_S). \tag{7}$$

In the following, we detail the justification of sampling methods only in the case of the Neumann crack problem, the case of the Dirichlet crack problem would be treated similarly.

It should be noticed that, strictly speaking, the Neumann crack problem is neither addressed in [11] nor in [3]. Let us now introduce some integral operators. We first define the hypersingular operator $T : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ by

$$T\phi(x) := \frac{\partial}{\partial \nu} \int_{\partial D} \phi(y) \frac{\partial G(x,y)}{\partial \nu(y)}\, ds(y), \quad x \in \partial D,$$

as well as its restrictions to γ, namely $T_\gamma : \tilde{H}^{\frac{1}{2}}(\gamma) \to H^{-\frac{1}{2}}(\gamma)$, with

$$T_\gamma \phi(x) := \frac{\partial}{\partial \nu} \int_{\gamma} \phi(y) \frac{\partial G(x,y)}{\partial \nu(y)}\, ds(y), \quad x \in \gamma. \tag{8}$$

We also define the auxiliary operator $G_N : H^{-\frac{1}{2}}(\gamma) \to L^2(\hat{S})$ which maps $g \in H^{-\frac{1}{2}}(\gamma)$ into the trace on $\hat{S}$ of the solution of the Neumann crack problem (4) with data g, as well as the integral operators $\mathcal{F}_N : \tilde{H}^{\frac{1}{2}}(\gamma) \to L^2(\hat{S})$ and $\mathcal{H}_N : L^2(\hat{S}) \to H^{-\frac{1}{2}}(\gamma)$ such that

$$(\mathcal{F}_N \phi)(x) := \int_{\gamma} \phi(y) \frac{\partial G(x,y)}{\partial \nu(y)}\, ds(y), \quad x \in \hat{S},$$

$$(\mathcal{H}_N h)(x) := \int_{\hat{S}} h(y) \frac{\partial G(x,y)}{\partial \nu(x)}\, ds(y), \quad x \in \gamma.$$

In order to prove some properties of the above operators, we need a unique continuation lemma, which is proved in [10].

Lemma 3.1. *For all $s > R$ and $h \in H^{\frac{1}{2}}(S_R)$, the following problem without the crack*

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad S \times (R,s) \\ \partial_\nu u = 0 & \text{on} \quad \Gamma \times (R,s) \\ u = h & \text{on} \quad S_R \\ \partial_\nu u = T_+ u & \text{on} \quad S_s \end{cases}$$

has a unique solution in $H^1(S \times (R,s))$, which is given by

$$u(x_S, x_3) = \sum_{n>0} (h, \theta_n)_{S_R} e^{i\beta_n (x_3 - R)} \theta_n(x_S).$$

We are now in a position to prove some useful properties of operators T_γ, $\mathcal{F}_N$, $\mathcal{H}_N$ and G_N. In this view we recall the following definitions: for some operator F,

$$\mathrm{Re} F = \frac{F + F^*}{2}, \quad \mathrm{Im} F = \frac{F - F^*}{2i},$$

where F^* denotes the adjoint of F, and for a selfadjoint operator F,

$$\text{if } F = \int_{-\infty}^{+\infty} \lambda\, dE_\lambda, \quad |F| := \int_{-\infty}^{+\infty} |\lambda|\, dE_\lambda,$$

where E_λ is the spectral family associated with the operator F.

Lemma 3.2. *The following assertions hold true under assumptions 2.1 and 2.4.*

(i) The operator T_γ is an isomorphism.

(ii) The operator $\mathrm{Im}T_\gamma$ *is non negative on* $\tilde{H}^{\frac{1}{2}}(\gamma)$*, that is*

$$\langle (\mathrm{Im}T_\gamma)\phi, \phi \rangle_{H^{-\frac{1}{2}}(\gamma), \tilde{H}^{\frac{1}{2}}(\gamma)} \geq 0, \quad \forall \phi \in \tilde{H}^{\frac{1}{2}}(\gamma).$$

(iii) The operators $\mathcal{F}_N$ *and* $\mathcal{H}_N$ *satisfy* $\mathcal{F}_N = \overline{\mathcal{H}_N^*}$*.*

(iv) The operators $\mathcal{F}_N$*,* G_N *and* T_γ *satisfy* $\mathcal{F}_N = G_N T_\gamma$*.*

(v) The operator G_N *is compact, injective with dense range.*

Proof. Let us consider the first assertion. The proof follows that of lemma 8.33 in [11]. We consider the operator T_∞ (resp. $T_{\gamma,\infty}$) the analogue of operator T (resp. T_γ) with the kernel G replaced by Φ, where Φ is the radiating Green's function of the Helmholtz equation in free space $\mathbb{R}^d$. The analogues of operators T_∞ and $T_{\gamma,\infty}$ in the special case $k = i$ are denoted $T_{\infty,i}$ and $T_{\gamma,\infty,i}$. For any $\phi \in \tilde{H}^{\frac{1}{2}}(\gamma)$, let us denote $\tilde{\phi}$ its extension by 0 in $H^{\frac{1}{2}}(\partial D)$. We hence have

$$\langle -T_{\gamma,\infty,i}\phi, \phi \rangle_{H^{-\frac{1}{2}}(\gamma), \tilde{H}^{\frac{1}{2}}(\gamma)} = \left\langle -T_{\infty,i}\tilde{\phi}, \tilde{\phi} \right\rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)}.$$

By theorem 1.26 in [17], $-T_{\infty,i}$ is a selfadjoint and coercive operator, so we have for some constant $c > 0$

$$\langle -T_{\gamma,\infty,i}\phi, \phi \rangle_{H^{-\frac{1}{2}}(\gamma), \tilde{H}^{\frac{1}{2}}(\gamma)} \geq c\,||\tilde{\phi}||^2_{H^{\frac{1}{2}}(\partial D)} = c\,||\phi||^2_{\tilde{H}^{\frac{1}{2}}(\gamma)},$$

which proves that $T_{\gamma,\infty,i}$ is an isomorphism. The operator $(T_\gamma - T_{\gamma,\infty,i})$ has a smooth kernel and therefore is compact. By using the decomposition $T_\gamma = T_{\gamma,\infty,i} + (T_\gamma - T_{\gamma,\infty,i})$, it is hence sufficient to prove that T_γ is injective.

Let us assume that for $\phi \in \tilde{H}^{\frac{1}{2}}(\gamma)$ we have $T_\gamma\phi = T\tilde{\phi} = 0$. We define the double layer potential

$$(\mathcal{D}\tilde{\phi})(x) := \int_{\partial D} \tilde{\phi}(y) \frac{\partial G(x,y)}{\partial \nu(y)}\, ds(y), \quad x \in W \setminus \partial D.$$

and the analogue operator $\mathcal{D}_\infty$ when the kernel G is replaced by Φ. Since the kernel of $(\mathcal{D} - \mathcal{D}_\infty)$ is smooth, the function $((\mathcal{D} - \mathcal{D}_\infty)\tilde{\phi})$ and its normal derivative are continuous across ∂D, which from classical jump relationship for double layer potential $\mathcal{D}_\infty$ (see for example [11]) implies

$$\tilde{\phi} = (\mathcal{D}_\infty\tilde{\phi})_+ - (\mathcal{D}_\infty\tilde{\phi})_- = (\mathcal{D}\tilde{\phi})_+ - (\mathcal{D}\tilde{\phi})_-,$$

$$T\tilde{\phi} = \partial_\nu(\mathcal{D}\tilde{\phi})_+ = \partial_\nu(\mathcal{D}\tilde{\phi})_-.$$

Since $T\tilde{\phi} = 0$, the function $(\mathcal{D}\tilde{\phi})$ solves the Neumann crack problem (4) with $g = 0$. Uniqueness for this problem implies that $(\mathcal{D}\tilde{\phi})$ vanishes in $W \setminus \overline{\gamma}$, which leads to $\tilde{\phi} = 0$ on ∂D, that is $\phi = 0$ on γ. This proves that T_γ is injective.

Let us consider the second assertion. For $\phi \in \tilde{H}^{\frac{1}{2}}(\gamma)$ let us denote $u := \mathcal{D}_\gamma \phi$, where the potential $\mathcal{D}_\gamma$ is given by

$$(\mathcal{D}_\gamma \phi)(x) := \int_\gamma \phi(y) \frac{\partial G(x,y)}{\partial \nu(y)}\, ds(y), \quad x \in W \setminus \overline{\gamma}. \tag{9}$$

First we notice that

$$\langle (\mathrm{Im} T_\gamma)\phi, \phi \rangle = \mathrm{Im}\, \langle T_\gamma \phi, \phi \rangle \,.$$

By using the weak formulation (5), we have

$$\langle T_\gamma \phi, \phi \rangle = \int_\gamma \partial_\nu u (\overline{u_+} - \overline{u_-})\, ds = -\int_{W_R} (|\nabla u|^2 - k^2 |u|^2)\, dx + \int_{S_{\pm R}} T_\pm u \overline{u}\, ds.$$

It comes from the definition of operators $T_\pm$ that

$$\int_{S_{\pm R}} T_\pm u \overline{u}\, ds = \sum_{n>0} i\beta_n |(u, \theta_n)_{S_{\pm R}}|^2,$$

which from (3) implies that

$$\mathrm{Im}\, \langle T_\gamma \phi, \phi \rangle = \sum_{n=1}^{n_p} \sqrt{k^2 - k_n^2} |(u, \theta_n)_{S_{\pm R}}|^2,$$

and the conclusion follows.
Let us consider the third assertion. To prove it, we write

$$(\mathcal{F}_N \phi, h)_{L^2(\hat{S})} = \int_{\hat{S}} \left(\int_\gamma \phi(x) \frac{\partial G(y,x)}{\partial \nu(x)}\, ds(x) \right) \overline{h(y)}\, dy$$

$$= \int_\gamma \phi(x) \left(\int_{\hat{S}} \frac{\partial G(y,x)}{\partial \nu(x)} \overline{h(y)}\, dy \right) ds(x).$$

Here we use the fact that $G(x,y) = G(y,x)$ for all $x, y \in W$ (in view of (7)). Then

$$(\mathcal{F}_N \phi, h)_{L^2(\hat{S})} = \int_\gamma \phi(x) \left(\int_{\hat{S}} \frac{\partial G(x,y)}{\partial \nu(x)} \overline{h(y)}\, dy \right) ds(x)$$

$$= \int_\gamma \phi(x) \overline{\left(\int_{\hat{S}} h(y) \frac{\overline{\partial G(x,y)}}{\partial \nu(x)}\, dy \right)}\, ds(x) = \left\langle \phi, \overline{\mathcal{H}_N h} \right\rangle_{\tilde{H}^{\frac{1}{2}}(\gamma), H^{-\frac{1}{2}}(\gamma)},$$

and the thesis follows.
The fourth assertion is obvious.
Let us consider the last assertion. That G_N is a compact operator results from the fact that the trace on $\hat{S}$ of the solution to the Neumann crack problem (4) belongs to $H^{\frac{1}{2}}(\hat{S})$, and the mapping $H^{\frac{1}{2}}(\hat{S}) \to L^2(\hat{S})$ is compact. Now let us prove injectivity. Assume that $G_N g = 0$ for some $g \in H^{-\frac{1}{2}}(\gamma)$ and let us denote by u the solution of the Neumann crack problem (4) associated with data g. Since the trace of u on S_R vanishes, from lemma 3.1 and unique

continuation it follows that u vanishes in $W \setminus \overline{\gamma}$, and then $g = 0$, which ends the proof. That G_N has dense range is equivalent to the fact that $\mathcal{F}_N$ has dense range from assertions (i) and (iv), and to the fact that $\mathcal{H}_N$ is injective from assertion (iii). Assume that $\mathcal{H}_N h = 0$ for some $h \in L^2(\hat{S})$. This implies that the function

$$v_h(x) := \int_{\hat{S}} h(y) G(x,y)\, ds(y), \quad x \in W_R \setminus \overline{\gamma}$$

solves the Neumann crack problem (4) with $g = 0$. Then $v_h = 0$ in $W_R \setminus \overline{\gamma}$. By using some decomposition $h = (h_-, h_+) \in L^2(S_{-R}) \times L^2(S_R)$ with $h_- = \sum_{n>0} h_n^- \theta_n$ and $h_+ = \sum_{n>0} h_n^+ \theta_n$ as well as the expression of the Green function given by (7), we obtain that for all x in $W_R \setminus \overline{\gamma}$

$$v_h(x) = -\sum_{n>0} \frac{h_n^-}{2i\beta_n} e^{i\beta_n (R+x_3)} \theta_n(x_S) - \sum_{n>0} \frac{h_n^+}{2i\beta_n} e^{i\beta_n (R-x_3)} \theta_n(x_S).$$

Since the θ_n form a transverse basis, we obtain that for all $n > 0$

$$h_n^- e^{i\beta_n x_3} + h_n^+ e^{-i\beta_n x_3} = 0$$

for an open interval of x_3. Given assumption 2.1 it follows that $h_n^- = h_n^+ = 0$ for all $n > 0$, and then $h = 0$, which completes the proof of the last assertion. □

3.2. The sampling methods

We now introduce the Linear Sampling Method and the Factorization Method. The subscript N refers to the Neumann crack problem, while D refers to the Dirichlet crack problem. We hence define the near field operators $F_N, \tilde{F}_N : L^2(\hat{S}) \to L^2(\hat{S})$ such that

$$(F_N h)(x) := \int_{\hat{S}} u_N^s(x,y) h(y)\, ds(y), \quad x \in \hat{S} \tag{10}$$

$$(\tilde{F}_N h)(x) := \int_{\hat{S}} \tilde{u}_N^s(x,y) h(y)\, ds(y), \quad x \in \hat{S}, \tag{11}$$

where $u_N^s(\cdot, y)$ and $\tilde{u}_N^s(\cdot, y)$ are the Neumann scattered fields associated with $u^i = G(\cdot, y)$ and $u^i = \overline{G(\cdot, y)}$, respectively, that is the solutions of the Neumann crack problem (4) with $g = -\partial_\nu G(\cdot, y)|_\gamma$ and $g = -\partial_\nu \overline{G(\cdot, y)}|_\gamma$, respectively.
We have the following factorization for the Neumann crack problem.

Proposition 3.3. *The near fields F_N and $\tilde{F}_N$ given by (10) and (11) have the factorization forms*

$$F_N = -G_N \overline{T_\gamma^*}\, \overline{G_N^*}, \quad \tilde{F}_N = -G_N T_\gamma^* G_N^*.$$

Proof. The proof immediately follows from the obvious identities

$$F_N = -G_N \mathcal{H}_N, \quad \tilde{F}_N = -G_N \overline{\mathcal{H}_N},$$

and on assertions (iii) and (iv) of lemma 3.2. □

The following proposition will enable us to choose the correct test function in the sampling formulations.

Proposition 3.4. *For some crack L, let us denote by $\mathcal{F}_N^L : \tilde{H}^{\frac{1}{2}}(L) \to L^2(\hat{S})$ the analogue of $\mathcal{F}_N$ when γ is replaced by L. For some continuous functions $\beta \in \tilde{H}^{\frac{1}{2}}(L)$ satisfying $|\beta| > 0$ on L, for the Neumann crack problem we have*

$$L \subset \gamma \quad \text{if and only if} \quad \mathcal{F}_N^L \beta \in \mathrm{R}(G_N).$$

Remark 3.5. *We should notice that existence of a function β defined as in the above proposition is not so easy. Such a function β shall vanish on ∂L in a suitable way (see theorem 1.2.16 in [15]).*

Proof. The proof is very similar to that of lemma 8.43 in [11]. Note that from lemma 3.2, since T_γ is an isomorphism and $\mathcal{F}_N = G_N T_\gamma$, we have that $\mathrm{R}(G_N) = \mathrm{R}(\mathcal{F}_N)$.
First, if $L \subset \gamma$, then since $\tilde{H}^{\frac{1}{2}}(L) \subset \tilde{H}^{\frac{1}{2}}(\gamma)$, we immediately have $\mathcal{F}_N^L \beta \in \mathrm{R}(\mathcal{F}_N)$.
Conversely, if $L \not\subset \gamma$, let us assume that $\mathcal{F}_N^L \beta = \mathcal{F}_N \phi$ for some $\phi \in \tilde{H}^{\frac{1}{2}}(\gamma)$. The two potentials $(\mathcal{D}_\gamma \phi)$ given by (9) and

$$\mathcal{D}_L(x) := \int_L \beta(y) \frac{\partial G(x,y)}{\partial \nu(y)}\, ds(y), \quad x \in W \setminus \overline{L}$$

both satisfy the system given in the statement of lemma 3.1 for $h := \mathcal{F}_N^L \beta = \mathcal{F}_N \phi$. By lemma 3.1 the functions $\mathcal{D}_L$ and $(\mathcal{D}_\gamma \phi)$ coincide on $S \times (R, +\infty)$, and by unique continuation they coincide on $W \setminus (\overline{\gamma} \cup \overline{L})$. Now let $x_0 \in L$ such that $x_0 \notin \overline{\gamma}$. Let $B(x_0, \varepsilon)$ an open ball such that $\overline{B(x_0,\varepsilon)} \cap \overline{\gamma} = \emptyset$ and $\overline{B(x_0,\varepsilon)} \cap \partial D_L \subset L$, where D_L is the closed manifold that contains L following the definition of crack L. The function $\mathcal{D}_\gamma \phi$ is smooth in $B(x_0, \varepsilon)$ while the function $\mathcal{D}_L$ is singular at the point x_0 since $|\beta| \geq c > 0$ on $\overline{B(x_0,\varepsilon)} \cap L$. Hence $L \subset \overline{\gamma}$, and since L and γ are open sets, $L \subset \gamma$. □

We are now in a position to state the main theorem that justifies the use of the Linear Sampling Method (part (i) of the theorem) and the Factorization method (part (ii) of the theorem) for the Neumann crack problem. The Linear Sampling Method (resp. Factorization Method) consists, for some test crack L, to check if the associated test function $\mathcal{F}_N^L \beta$ belongs or not to the range of operator F_N (resp. $\tilde{F}_N^{\diamond}$), the both operators depending only on the data. In the present paper, we have chosen to characterize such a range test with the help of the Tikhonov regularization.
In this view, for a linear bounded operator $F : V \to V$, let us consider the operator $T_\varepsilon(F) : V \to V$ associated with the Tikhonov regularization of operator F for parameter $\varepsilon > 0$, namely

$$T_\varepsilon(F) := (\varepsilon I + F^* F)^{-1} F^*,$$

where I is the identity operator on V. In addition, for a selfadjoint operator F, we consider the selfadjoint operator $F^{\diamondsuit} : V \to V$, defined by

$$F^{\diamondsuit} := \sqrt{|\mathrm{Re}F| + |\mathrm{Im}F|}.$$

Theorem 3.6. *Let $F_N : L^2(\hat{S}) \to L^2(\hat{S})$ and $\tilde{F}_N : L^2(\hat{S}) \to L^2(\hat{S})$ be the near field operators defined by (10) and (11) with $u_N^s(\cdot, y)$ and $\tilde{u}_N^s(\cdot, y)$ being the solution of the Neumann crack problem (4) with $g = -\partial_\nu G(\cdot, y)|_\gamma$ and $g = -\partial_\nu \overline{G(\cdot, y)}|_\gamma$ for $y \in \hat{S}$, respectively. Let us define, for some continuous function $\beta \in \tilde{H}^{\frac{1}{2}}(\gamma)$ with $|\beta| > 0$ on L, the test function*

$$(\mathcal{F}_N^L \beta)(x) := \int_L \beta(y) \frac{\partial G(x, y)}{\partial \nu(y)}\, ds(y), \quad x \in \hat{S}.$$

For

$$h_\varepsilon := T_\varepsilon(F_N)(\mathcal{F}_N^L \beta) \quad \text{and} \quad \tilde{h}_\varepsilon := T_\varepsilon(\tilde{F}_N^{\diamondsuit})(\mathcal{F}_N^L \beta),$$

(i) $\qquad L \not\subset \gamma \quad \text{implies that} \quad \lim_{\varepsilon \to 0} ||h_\varepsilon||_{L^2(\hat{S})} = +\infty.$

(ii) $\qquad L \subset \gamma \quad \text{if and only if} \quad \lim_{\varepsilon \to 0} ||\tilde{h}_\varepsilon||_{L^2(\hat{S})} < \infty.$

To prove the second part of the above theorem, we will need the following abstract theorem, which is proved in [16, 19] with weaker assumptions.

Theorem 3.7. *Let $X \subset U \subset X^*$ be some Hilbert spaces such that each embedding is dense. Furthermore, let V be another Hilbert space which we identify to its dual V^*, and $F : V \to V$, $H : V \to X$ and $T : X \to X^*$ be linear and bounded operators with $F = H^* T H$. We make the following assumptions:*

(i) *H is compact and injective.*

(ii) ReT *has the form* Re$T = T_0 + T_1$ *with some selfadjoint and coercive operator T_0 and some compact operator $T_1 : X \to X^*$.*

(iii) ImT *is non negative on X, that is $\langle (\mathrm{Im}T)\phi, \phi \rangle \geq 0$, for all $\phi \in X$.*

(iv) *T is injective.*

Then the operator $F^{\diamondsuit}$ is a selfadjoint, positive operator and the ranges of $H^* : X^* \to V$ and $F^{\diamondsuit} : V \to V$ coincide.

Proof of theorem 3.6. We first recall the following standard result concerning the Tikhonov regularization of a compact operator $F : V \to V$ which is injective and has dense range (see for example [20]): $\lim_{\varepsilon \to 0} ||T_\varepsilon(F)(f)||_V$ exists for all $f \in V$ and

$$f \in \mathrm{R}(F) \quad \text{iff} \quad \lim_{\varepsilon \to 0} ||T_\varepsilon(F)(f)||_V < +\infty. \tag{12}$$

Let us prove the first assertion. From proposition 3.4, $L \not\subset \gamma$ implies that $\mathcal{F}_N^L \beta \notin \mathrm{R}(G_N)$, and from proposition 3.3 that $\mathcal{F}_N^L \beta \notin \mathrm{R}(F_N)$. We remark from lemma 3.2 and proposition 3.3 that F_N is compact, injective with dense range. We hence complete the proof by using

(12) for $F = F_N$.
Let us prove the second assertion. The combination of proposition 3.3, proposition 3.4 and theorem 3.7 in the particular case $U = L^2(\gamma)$, $X = \tilde{H}^{\frac{1}{2}}(\gamma)$, $V = L^2(\hat{S})$, $F = \tilde{F}_N$, $H = G_N^*$ and $T = -T_\gamma^*$ proves that

$$L \subset \gamma \quad \text{iff} \quad f \in \mathrm{R}(\tilde{F}_N^{\diamondsuit}).$$

We simply have to verify the four assumptions of theorem 3.7. The assumption (i) is justified by the last statement of lemma 3.2. By using the decomposition $T_\gamma = T_{\gamma,\infty,i} + (T_\gamma - T_{\gamma,\infty,i})$ already used in the proof of lemma 3.2, we have $\mathrm{Re}\,(-T_\gamma^*) = -T_{\gamma,\infty,i} - \mathrm{Re}\,(T_\gamma - T_{\gamma,\infty,i})$, where the selfadjoint operator $T_{\gamma,\infty,i}$ satisfies $\langle -T_{\gamma,\infty,i}\phi, \phi\rangle \geq c\,||\phi||^2_{\tilde{H}^{\frac{1}{2}}(\gamma)}$ for all $\phi \in \tilde{H}^{\frac{1}{2}}(\gamma)$ and $\mathrm{Re}\,(T_\gamma - T_{\gamma,\infty,i})$ is compact. Assumption (ii) is then satisfied. The assumption (iii) follows from the second assertion of lemma 3.2:

$$\big\langle (\mathrm{Im}(-T_\gamma^*))\phi, \phi\big\rangle = \langle (\mathrm{Im}T_\gamma)\phi, \phi\rangle \geq 0.$$

Lastly, assumption (iv) is a consequence of the first statement of lemma 3.2. We complete the proof by using (12) for $F = \tilde{F}_N^{\diamondsuit}$. □

Remark 3.8. *It should be noticed that the justification is rigorous concerning the Factorization Method, while it is only partial concerning the Linear Sampling Method, since nothing is said about the case $L \subset \gamma$. What we can prove is that if $L \subset \gamma$, then for all $\varepsilon > 0$ there exists a solution $h_\varepsilon \in L^2(\hat{S})$ of the inequality $||F_N h_\varepsilon - \mathcal{F}_N^L\beta||_{L^2(\hat{S})} \leq \varepsilon$ such that the function $\mathcal{H}_N h_\varepsilon$ converges in $H^{-\frac{1}{2}}(\gamma)$ as $\varepsilon \to 0$.*

Concerning the Dirichlet crack problem, with the same analysis as above we can establish a similar theorem as 3.6 but we shall consider the near fields operator F_D and $\tilde{F}_D$ associated with the solutions $u_D^s(\cdot, y)$ and $\tilde{u}_D^s(\cdot, y)$ of the Dirichlet crack problem (4) with $f = -G(\cdot, y)|_\gamma$ and $f = -\overline{G(\cdot, y)}|_\gamma$ for $y \in \hat{S}$, respectively, and as in [11] we shall use the test function defined by

$$(\mathcal{F}_D^L\alpha)(x) := \int_L \alpha(y)G(x, y)\, ds(y), \quad x \in \hat{S},$$

for some continuous function $\alpha \in \tilde{H}^{-\frac{1}{2}}(\gamma)$ with $|\alpha| > 0$ on L.
Note that in such case the isomorphism T_γ involved in the proposition 3.3 is replaced by the isomorphism $S_\gamma : \tilde{H}^{-\frac{1}{2}}(\gamma) \to H^{\frac{1}{2}}(\gamma)$ defined by

$$(S_\gamma\psi)(x) := \int_\gamma \psi(y)G(x, y)\, ds(y), \quad x \in \gamma.$$

Remark 3.9. *It results from the above analysis that when the boundary condition on the crack is a priori known to be of Dirichlet type or a priori known to be of Neumann type, the test function used in the sampling method has to be adjusted to such boundary condition: it should be what we will call the Dirichlet test function $\mathcal{F}_D^L\alpha$ for the Dirichlet crack, while* *it should be the Neumann test function $\mathcal{F}_N^L \beta$ for the Neumann crack. We will see in the numerical experiments that the right choice of such test function is very important to obtain a correct image of the crack. We should also note that when the boundary condition is unknown or in the case of an impedance boundary condition like in [11, 3], both the Dirichlet and Neumann test functions have to be used simultaneously.*

For sake of convenience, in the following we only test infinitesimal cracks L at point $z \in W$ oriented by normal $\nu(z)$, with $\int_L \alpha\, ds = \int_L \beta\, ds = 1$, so that we make the approximations

$$\mathcal{F}_D^L \alpha \simeq f_D^z := G(\cdot, z), \quad \mathcal{F}_N^L \beta \simeq f_N^z := \nabla_z G(\cdot, z) \cdot \nu(z). \tag{13}$$

Hence, in the case of the Dirichlet crack problem, the Linear Sampling Method (resp. Factorization Method) consists in computing, for all sampling points $z \in W_R$ and small parameter ε, the $L^2(\hat{S})$ function $h_\varepsilon = T_\varepsilon(F_D)(f_D^z)$ (resp. $\tilde{h}_\varepsilon = T_\varepsilon(\tilde{F}_D^{\diamondsuit})(f_D^z)$), and then in plotting $1/||h_\varepsilon||_{L^2(\hat{S})}$ (resp. $1/||\tilde{h}_\varepsilon||_{L^2(\hat{S})}$) as a function of z. Following theorem 3.6, such function vanishes in the complementary domain of the crack.

The case of the Neumann crack problem is more complicated since the unit normal to the crack $\nu(z)$ is unknown. Let us consider the case of the Linear Sampling Method. Following the idea introduced in [3], for all sampling points $z \in W_R$, we replace $\nu(z)$ in the definition of f_N^z by a polarization vector p. For all $p = e_i$, $i = 1, 2, 3$, we compute the $L^2(\hat{S})$ functions $h_{\varepsilon,i} = T_\varepsilon(F_N)(f_{N,i}^z)$ with $f_{N,i}^z := \nabla_z G(\cdot, z) \cdot e_i$ and lastly use the decomposition $\nu(z) = p_1\, e_1 + p_2\, e_2 + p_3\, e_3$ and the linearity of the operator $T_\varepsilon(F_N)$ to obtain

$$h_\varepsilon = p_1\, h_{\varepsilon,1} + p_2\, h_{\varepsilon,2} + p_3\, h_{\varepsilon,3}. \tag{14}$$

In the spirit of theorem 3.6, the vector (p_1, p_2, p_3) is computed by minimizing $||h_\varepsilon||_{L^2(\hat{S})}$ with constraint $\sqrt{p_1^2 + p_2^2 + p_3^2} = 1$. As for the Dirichlet case, we finally plot $1/||h_\varepsilon||_{L^2(\hat{S})}$ as a function of z with polarization p set to the obtained optimal value at point z. Of course, the same process can be applied to the Factorization Method.

3.3. The modal formulation

Let us go back to our inverse problem (IP), which requires to express the functions $u_D^s(\cdot, y)$, $u_N^s(\cdot, y)$, $\tilde{u}_D^s(\cdot, y)$, $\tilde{u}_N^s(\cdot, y)$ for $y \in \hat{S}$ only in terms of the scattered fields $u_n^\pm$ which are associated with the incident waves formed by the guided modes $g_n^\pm$ for the Dirichlet or Neumann crack problem. Whenever it is possible, the subscripts D and N will be omitted hereafter in order to shorten notations.

In the following lemma, we first give an expression of the Green function (7) and its complex conjugate only in terms of the guided modes $g_n^\pm$.

Lemma 3.10. *For all $x \in W_R$, we have*

$$G(x,y) = \begin{cases} -\displaystyle\sum_{n>0} \frac{1}{2i\beta_n} g_n^+(x) g_n^-(y), & \forall y \in S_{-R}, \\ -\displaystyle\sum_{n>0} \frac{1}{2i\beta_n} g_n^-(x) g_n^+(y), & \forall y \in S_R, \end{cases}$$

$$\overline{G(x,y)} = \begin{cases} \displaystyle\sum_{0<n\le n_p} \frac{1}{2i\beta_n} g_n^-(x) g_n^+(y) - \sum_{n>n_p} \frac{1}{2i\beta_n} g_n^+(x) g_n^-(y), & \forall y \in S_{-R}, \\ \displaystyle\sum_{0<n\le n_p} \frac{1}{2i\beta_n} g_n^+(x) g_n^-(y) - \sum_{n>n_p} \frac{1}{2i\beta_n} g_n^-(x) g_n^+(y), & \forall y \in S_R. \end{cases}$$

The proof is straightforward and results from the expression of the β_n (3). We can see that taking the conjugate of the Green function has no effect on the evanescent part of the sum while, up to a change of sign, it interchanges the role of the propagating modes traveling from the left to the right and the propagating modes traveling from the right to the left. The previous lemma implies that if $u^s(\cdot,y)$ and $\tilde{u}^s(\cdot,y)$ denote the scattered field associated with the incident waves $G(\cdot,y)$ and $\overline{G(\cdot,y)}$, by linearity we immediately obtain the

Proposition 3.11. *For all $x \in W_R$, we have*

$$u^s(x,y) = \begin{cases} -\displaystyle\sum_{n>0} \frac{1}{2i\beta_n} u_n^+(x) g_n^-(y), & \forall y \in S_{-R}, \\ -\displaystyle\sum_{n>0} \frac{1}{2i\beta_n} u_n^-(x) g_n^+(y), & \forall y \in S_R, \end{cases}$$

$$\tilde{u}^s(x,y) = \begin{cases} \displaystyle\sum_{0<n\le n_p} \frac{1}{2i\beta_n} u_n^-(x) g_n^+(y) - \sum_{n>n_p} \frac{1}{2i\beta_n} u_n^+(x) g_n^-(y), & \forall y \in S_{-R}, \\ \displaystyle\sum_{0<n\le n_p} \frac{1}{2i\beta_n} u_n^+(x) g_n^-(y) - \sum_{n>n_p} \frac{1}{2i\beta_n} u_n^-(x) g_n^+(y), & \forall y \in S_R. \end{cases}$$

From proposition 3.11, we are now able to derive some explicit expressions of near field operators F and $\tilde{F}$ (for either Dirichlet or Neumann crack problems) in terms of the data $u_n^\pm|_{\hat{S}}$. In this view we use the following decomposition of such data in the transverse basis θ_m of $L^2(S_{\pm R})$.

$$u_n^+|_{S_{-R}} = \sum_{m>0} (U_n^+)_m^- \,\theta_m, \quad u_n^-|_{S_{-R}} = \sum_{m>0} (U_n^-)_m^- \,\theta_m,$$

$$u_n^+|_{S_R} = \sum_{m>0} (U_n^+)_m^+ \,\theta_m, \quad u_n^-|_{S_R} = \sum_{m>0} (U_n^-)_m^+ \,\theta_m.$$

Using then the decomposition $h = (h^-, h^+) \in S_{-R} \times S_R$ with

$$h^- = \sum_{m>0} h_m^- \,\theta_m, \quad h^+ = \sum_{m>0} h_m^+ \,\theta_m,$$

we obtain after easy computations (see [10]) the following expression of operators F and $\tilde{F}$:

$$\begin{cases} (Fh)|_{S_{-R}} = -\displaystyle\sum_{m>0}\sum_{n>0} \frac{e^{i\beta_n R}}{2i\beta_n}\left((U_n^+)_m^-\, h_n^- + (U_n^-)_m^-\, h_n^+\right)\theta_m, \\ (Fh)|_{S_R} = -\displaystyle\sum_{m>0}\sum_{n>0} \frac{e^{i\beta_n R}}{2i\beta_n}\left((U_n^+)_m^+\, h_n^- + (U_n^-)_m^+\, h_n^+\right)\theta_m, \end{cases}$$

$$\begin{cases} (\tilde{F}h)|_{S_{-R}} = \displaystyle\sum_{m>0}\sum_{0<n\le n_p} \frac{e^{-i\beta_n R}}{2i\beta_n}\left((U_n^-)_m^-\, h_n^- + (U_n^+)_m^-\, h_n^+\right)\theta_m \\ \qquad\qquad -\displaystyle\sum_{m>0}\sum_{n>n_p} \frac{e^{i\beta_n R}}{2i\beta_n}\left((U_n^+)_m^-\, h_n^- + (U_n^-)_m^-\, h_n^+\right)\theta_m, \\ (\tilde{F}h)|_{S_R} = \displaystyle\sum_{m>0}\sum_{0<n\le n_p} \frac{e^{-i\beta_n R}}{2i\beta_n}\left((U_n^-)_m^+\, h_n^- + (U_n^+)_m^+\, h_n^+\right)\theta_m \\ \qquad\qquad -\displaystyle\sum_{m>0}\sum_{n>n_p} \frac{e^{i\beta_n R}}{2i\beta_n}\left((U_n^+)_m^+\, h_n^- + (U_n^-)_m^+\, h_n^+\right)\theta_m. \end{cases}$$

Similarly, we obtain from lemma 3.10 the following expansions of the test functions f_D^z and f_N^z:

$$\begin{cases} f_D^z|_{S_{-R}} = -\displaystyle\sum_{m>0}\left(\frac{e^{i\beta_m(R+z_3)}}{2i\beta_m}\theta_m(z_S)\right)\theta_m, \\ f_D^z|_{S_R} = -\displaystyle\sum_{m>0}\left(\frac{e^{i\beta_m(R-z_3)}}{2i\beta_m}\theta_m(z_S)\right)\theta_m. \end{cases}$$

$$\begin{cases} f_N^z|_{S_{-R}} = -\displaystyle\sum_{m>0}\left(\frac{e^{i\beta_m(R+z_3)}}{2i\beta_m}(\nabla_S\theta_m(z_S)\cdot\nu_S(z) + i\beta_n\theta_m(z_S)\nu_3(z))\right)\theta_m, \\ f_N^z|_{S_R} = -\displaystyle\sum_{m>0}\left(\frac{e^{i\beta_m(R-z_3)}}{2i\beta_m}(\nabla_S\theta_m(z_S)\cdot\nu_S(z) - i\beta_n\theta_m(z_S)\nu_3(z))\right)\theta_m, \end{cases}$$

where $\nabla_S\cdot$ denotes the surface gradient in a transverse section. The far field formulation consists then to restrict the sums that are involved in the definition of near field operators F and $\tilde{F}$ to the n_p first terms, since n_p is the number of propagating modes, as well as in the sum involved in the definition of the test functions f_D^z and f_N^z. We hence drop the data associated with the evanescent modes. The influence of such evanescent modes on the quality of identification is studied in [10] for soft impenetrable obstacles.

3.4. Some numerical experiments

In our numerical experiments, we consider a 2D waveguide of section $S = (-1, 1)$, and we apply both the Linear Sampling Method and the Factorization Method in the particular case of the modal far field formulation, the number of propagating modes n_p being given as a function of the wavenumber k. In practice, the data $u_n^\pm|_{S_{-R}}$ and $u_n^\pm|_{S_R}$ are contaminated by some noise of amplitude δ, and the parameter ε in the Tikhonov regularization is chosen as a function of δ following the Morozov's strategy introduced in [14] in the framework of the Linear Sampling Method. In our modal far field formulation, we exactly use the same Morozov's technique as in [10] to choose ε. The synthetic data $u_n^\pm$ are obtained by using a finite element approximation of the weak formulation (5) of problem (4). Precisely, we used classical Lagrange triangles based on a mesh which is sufficiently refined to be acceptable for the larger wavenumber k we consider in our numerical experiments. The artificial noisy data are produced by applying to each exact data $\hat{u} = u_n^\pm|_{\hat{S}}$ for $n = 1, \cdots, n_p$ some pointwise Gaussian noise which is then calibrated in order to obtain some noisy data $\hat{u}^\delta$ satisfying

$$||\hat{u}^\delta - \hat{u}||_{L^2(\hat{S})} = \delta = \sigma||\hat{u}||_{L^2(\hat{S})},$$

where σ is some prescribed relative amplitude of noise. In figure 1, we show the results obtained by using the Factorization Method (based on $\tilde{F}$), in the case of a Dirichlet or a Neumann curved crack. The Linear Sampling Method produces similar results. Here we have used $n_p = 16$ propagating modes and exact data. In both cases, we apply the Dirichlet test function f_D^z and the Neumann test function f_N^z, and in this last case we apply our optimization technique to find the local normal to the crack. Note that in our $2D$ case, the decomposition (14) amounts to $h_\varepsilon = (\cos\theta)\, h_{\varepsilon,1} + (\sin\theta)\, h_{\varepsilon,3}$, where $(\pi/2 - \theta)$ is the angle between the direction of propagation e_3 and the polarization vector p. The optimal value of θ is obtained by an elementary calculation. In the figure 1, the polarization vector p associated with such optimal θ is represented on the crack. Those results emphasize the fact that for a Dirichlet crack problem, a Dirichlet test function has to be used, and similarly for a Neumann crack problem, a Neumann test function has to be used. We obtain poor results in the other cases.

Remark 3.12. *We remark that the result for identification is better for the Dirichlet crack and the Neumann test function than for the Neumann crack and the Dirichlet test function. This is due to the fact that for small crack L at point z, by a first order Taylor expansion we obtain*

$$\mathcal{F}_D^L\beta = \int_L \alpha(y)G(\cdot, y)\, ds(y) \approx \int_L \left(\alpha(y)G(\cdot, z) + \alpha(y)\frac{\partial G(\cdot, z)}{\partial\tau(z)}(y - z)\right) ds(y),$$

where τ is the unit tangent vector to the crack, and lastly

$$\mathcal{F}_D^L\beta \approx G(\cdot, z) + \alpha_1(z)\nabla_z G(\cdot, z)\cdot\tau(z), \quad \alpha_1(z) := \int_L \alpha(y)(y - z)\, ds(y).$$

As a result, the Neumann test function may be used for the Dirichlet crack, but the optimal polarization $p(z)$ coincides with the tangent unit vector $\tau(z)$ instead of the normal unit vector $\nu(z)$, as confirmed on the top right part of figure 1.

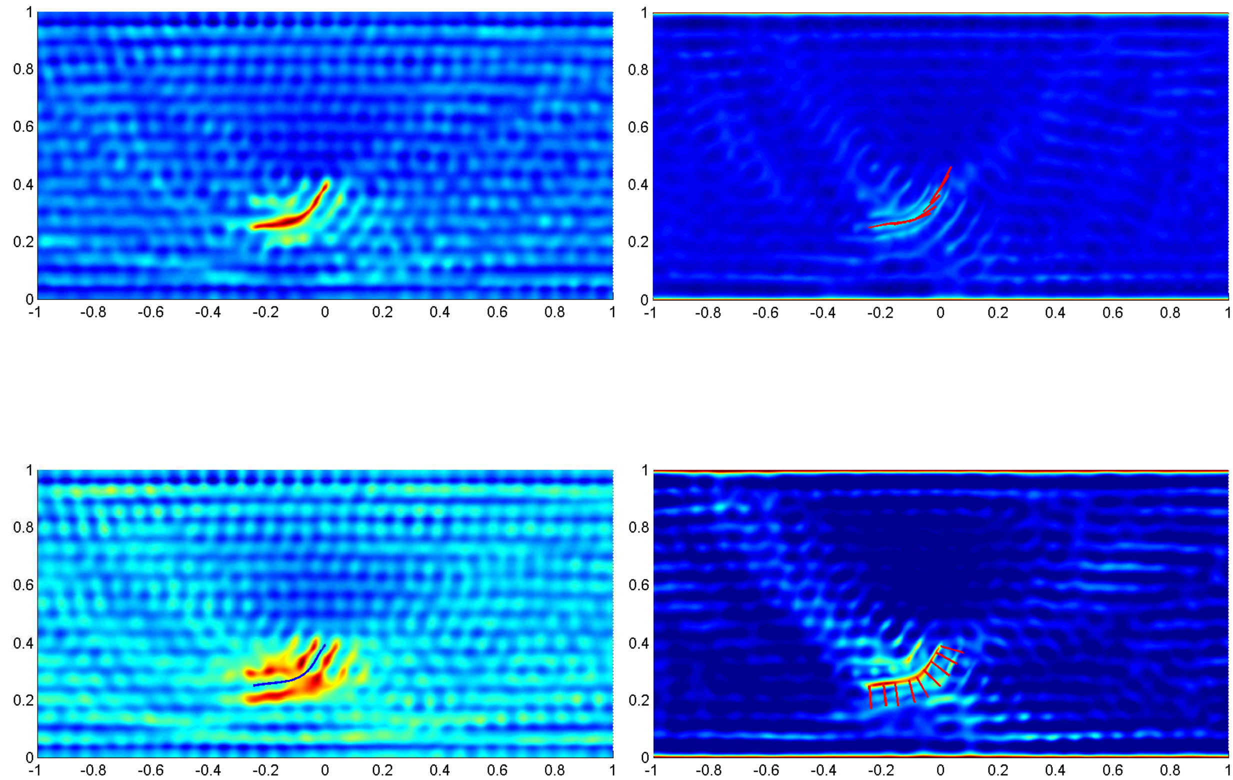


Figure 1. Top left: Dirichlet crack and Dirichlet test function, Top right: Dirichlet crack and Neumann test function (optimal polarization is represented in red), Bottom left: Neumann crack and Dirichlet test function (the real crack is represented in blue), Bottom right: Neumann crack and Neumann test function (optimal polarization is represented in red).

In figure 2 we study the influence of the frequency, that is in other words, the influence of the number of propagating modes. The data are free of noise. We only consider the case of the Linear Sampling Method (results obtained with the Factorization Method are similar) for a set of two curved Dirichlet cracks, and for the suitable Dirichlet test function. The results of figure 2 correspond to $n_p = 4$, $n_p = 10$, $n_p = 16$ and $n_p = 23$ and show that the higher is n_p, the better is the result with a saturation effect for too high n_p, which is due to the discretization.

In figure 3 we study the influence of the amplitude of noise, for a given number of propagating modes $n_p = 16$. We only consider the case of the Linear Sampling Method for a set of two curved Neumann cracks, and for the suitable Neumann test function with optimal polarization. The results of figure 3 correspond to $\sigma = 0.01$, $\sigma = 0.1$, $\sigma = 0.2$ and $\sigma = 0.5$.

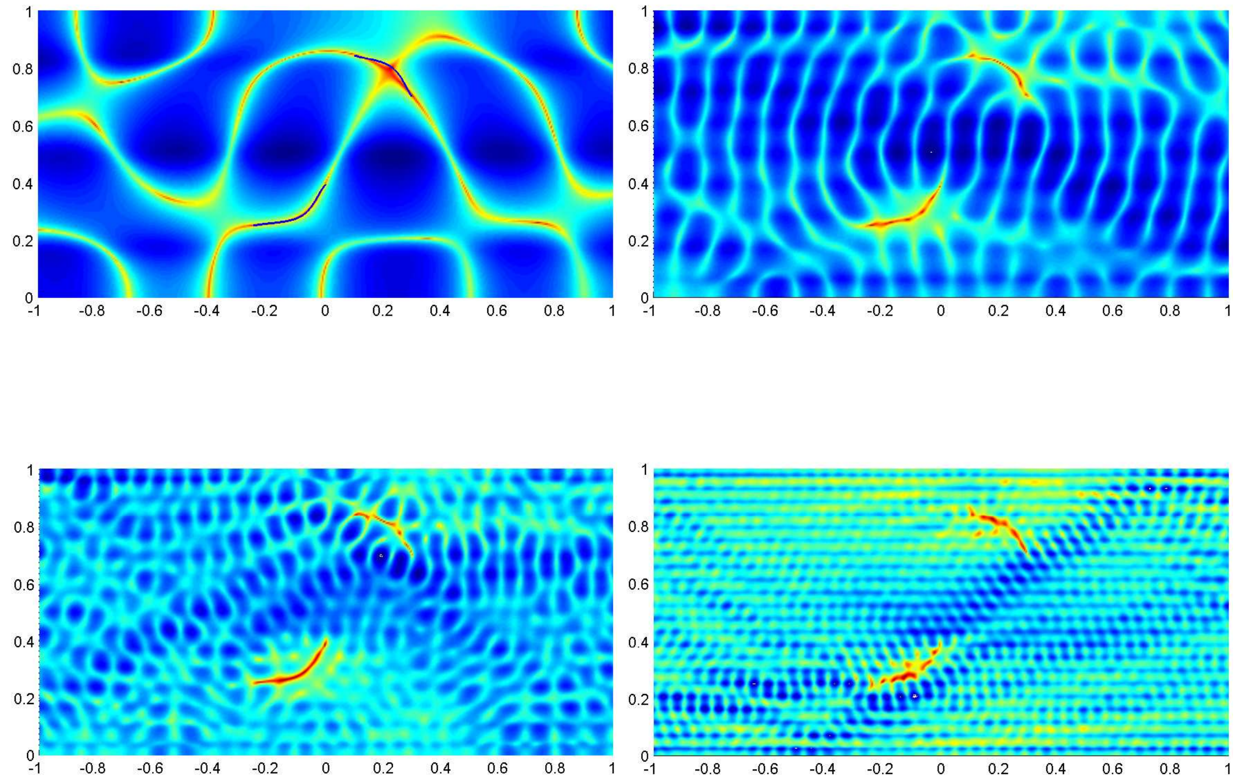


Figure 2. Top left: $n_p = 4$ (the real cracks are represented in blue), Top right: $n_p = 10$, Bottom left : $n_p = 16$, Bottom right: $n_p = 23$

They show that the sampling methods are very robust with respect to a Gaussian noise. It may be surprising that, even with a 50%-noise level, the reconstruction be satisfactory. This is due to the way we produce the noise: the amplitude of noise is large with respect to the L^2 norm but the noisy data is strongly oscillating around the exact data, so that its projection on the modes tends to smooth them a lot. The reader can see in [10] an example of noisy data produced by our method compared to the exact one.

We complete the numerical section by a comparison on figure 4 between the Linear Sampling Method and the Factorization Method for a set of two Dirichlet or Neumann cracks that are close to each other. This is a complex situation since waves are likely to be trapped between these two cracks and therefore the sampling methods cannot easily separate the two cracks. The data are free of noise and $n_p = 16$. We conclude from figure 4 that from a numerical point of view, the quality of the identification produced by the two sampling methods are approximately the same, even if the theoretical justification is rigorous only in the case of the Factorization Method (see remark 3.8).

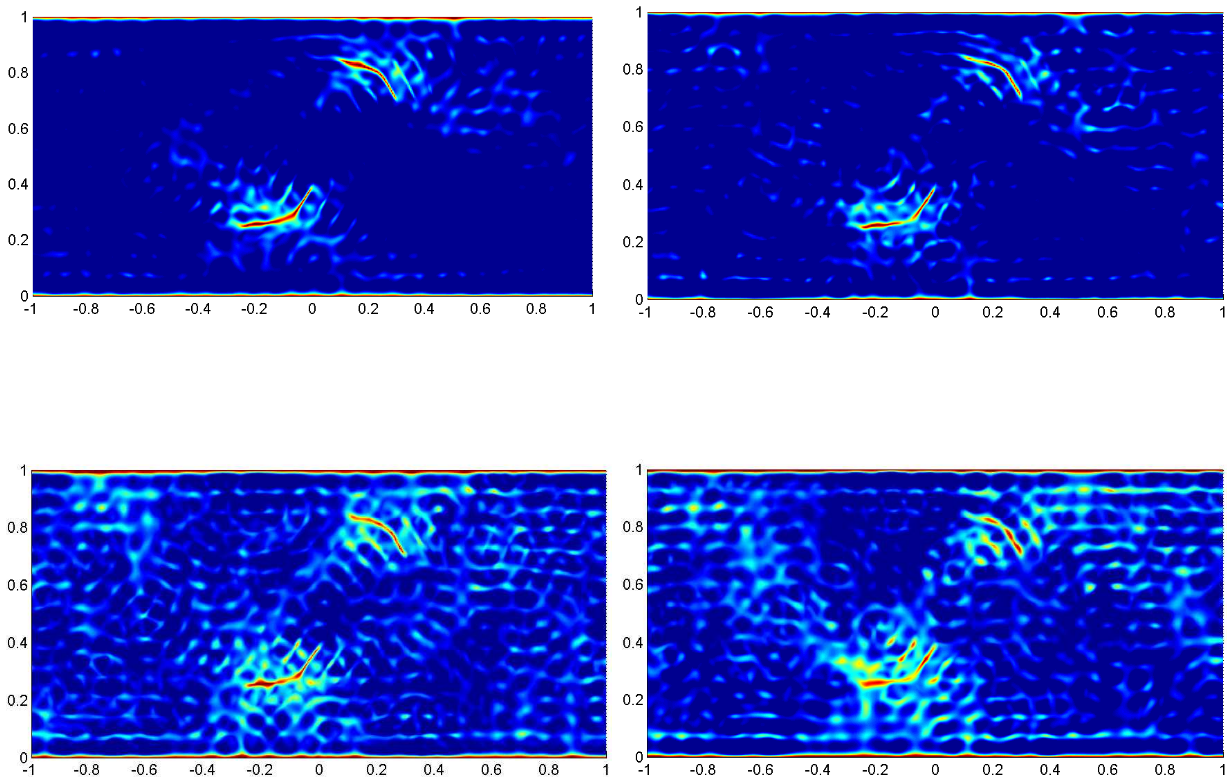


Figure 3. Top left: $\sigma = 0.01$, Top right : $\sigma = 0.1$, Bottom left: $\sigma = 0.2$, Bottom right : $\sigma = 0.5$

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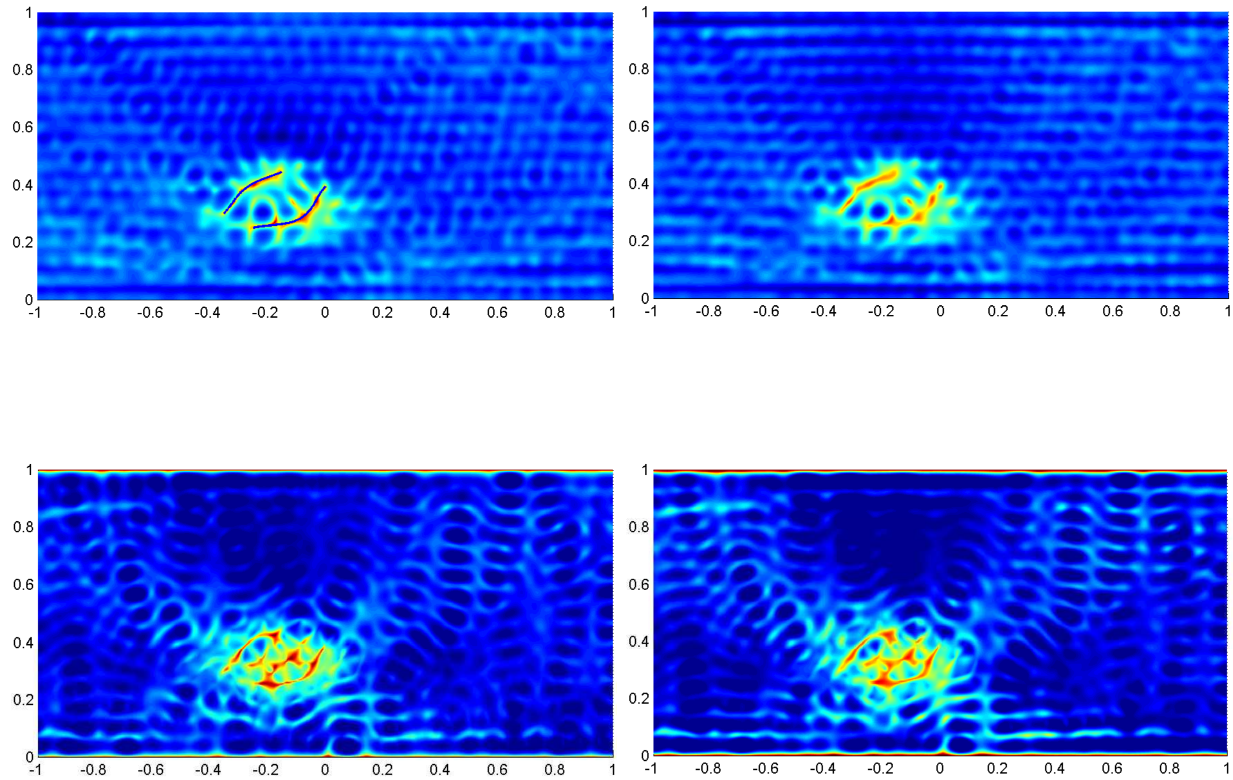

Figure 4. Top left: LSM with Dirichlet crack, Top right: FM with Dirichlet crack, Bottom left: LSM with Neumann crack, Bottom right: FM with Neumann crack

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