Using time reversal for space–time focusing of acoustic waves

Maxence Cassier, Christophe Hazard and Patrick Joly

POEMS (UMR 7231 CNRS-ENSTA-INRIA),
32 Boulevard Victor,
75015 Paris,
France
christophe.hazard@ensta-paristech.fr

ABSTRACT. In this paper, we are concerned with focusing effects for time-dependent waves using 'time reversal mirrors'. We consider a simple two-dimensional problem which models acoutic wave propagation in a homogeneous medium which contains several unknown scatterers. We show how to construct a wave that focuses in space and time near one of these scatterers, in the form of a superposition of time-harmonic waves related to the eigenvectors of the so-called 'time reversal operator'.

RÉSUMÉ. Dans cette communication, on s'intéresse aux effets de focalisation obtenus à l'aide d'un 'miroir à retournement temporel'. On considère un problème bi-dimensionnel simple qui modélise la propagation d'ondes acoustiques dans un milieu homogène contenant quelques diffuseurs inconnus. On montre comment construire une onde qui focalise en espace et en temps au voisinage d'un de ces diffuseurs, sous la forme d'une superposition d'ondes périodiques en temps, ces dernières étant reliées aux vecteurs propres d'un opérateur dit de retournement temporel.

KEYWORDS : Focusing, Time reversal, DORT method, Small scatterers

MOTS-CLÉS : Focalisation, Retournement temporel, Méthode DORT, Petits diffuseurs

1. Introduction

The present paper is motivated by the following challenging question: in a propagative medium which contains several unknown scatterers, how can one generate a wave that focuses selectively on one scatterer not only in space, but also in time, in other words, a wave that 'hits hard at the right spot'? Such focusing properties have been studied in the frequency domain in the context of the DORT method (French acronym for "Decomposition of the Time Reversal Operator", see, e.g., [4, 5]). In short, a array of transducers, called here a Time Reversal Mirror (TRM), first emits a wave which propagates in the medium and interacts with the scatterers. In a second step, the TRM measures the scattered wave, and time-reverses this measure, so that it can re-emit a wave in a last-in first-out process. The operator which represents two successive iterations of this loop is referred to as the Time Reversal Operator (TRO). It is now understood that for small and distant enough scatterers, one can choose an eigenvector of the TRO such that the corresponding input signals generate a wave which focuses selectively on one scatterer. Can we take advantage of these spatial focusing properties which hold for time-harmonic waves to produce a time-dependent wave which would be also focused in the time domain? A natural idea could be to iterate the above mentioned loop in the time-domain. But it is shown in [1] that such an iterative process leads in general exactly to the opposite of time-focusing: it produces a time-harmonic wave! In the present paper, we show how to use the eigenelements of the TRO to produce space-time focusing.

For the sake of simplicity, we deal with a two-dimensional acoustic problem. We consider a family of P sound-soft circular scatterers $\mathcal{O}_p$ for $p = 1, \dots, P$, of respective radii $r_1, \dots, r_P$ and centers $s_1, \dots, s_P$, located in a homogeneous medium filling the whole plane $\mathbb{R}^2$. We assume that the TRM is composed of N pointlike transducers x_n for $n = 1, \dots, N$. The question is to find the input signals to be send to the traducers so that they generate a wave that focuses on one of the scatterers. This wave $\mathcal{U} = \mathcal{U}(x,t)$ is solution to

$$\frac{\partial^2 \mathcal{U}}{\partial t^2} - \Delta \mathcal{U} = \sum_{n=1}^{N} \mathcal{F}_n \otimes \delta_{x_n} \quad \text{in } \mathbb{R}^2 \setminus \textstyle\bigcup_{p=1}^{P} \mathcal{O}_p,$$

$$\mathcal{U} = 0 \quad \text{on } \textstyle\bigcup_{p=1}^{P} \partial\mathcal{O}_p,$$

where $\mathcal{F}_1(t), \dots, \mathcal{F}_N(t)$ denote the input signals and δ_{x_n} is the Dirac measure at x_n. Function $\mathcal{U}$ can be decomposed as $\mathcal{U} = \mathcal{V} + \mathcal{W}$ where $\mathcal{W}$ represents the *incident* wave, solution to

$$\frac{\partial^2 \mathcal{W}}{\partial t^2} - \Delta \mathcal{W} = \sum_{n=1}^{N} \mathcal{F}_n \otimes \delta_{x_n} \quad \text{in } \mathbb{R}^2,$$

whereas $\mathcal{V}$ stands for the *scattered* wave, solution to

$$\frac{\partial^2 \mathcal{V}}{\partial t^2} - \Delta \mathcal{V} = 0 \quad \text{in } \mathbb{R}^2 \setminus \textstyle\bigcup_{p=1}^{P} \mathcal{O}_p, \tag{1}$$

$$\mathcal{V} = -\mathcal{W} \quad \text{on } \textstyle\bigcup_{p=1}^{P} \partial\mathcal{O}_p. \tag{2}$$

As mentioned above, the idea is to construct the input signals by means of the eigenelements of the TRO, which is related to the time-harmonic problem associated with (1)–(2). For a given frequency ω and a time-harmonic incident wave $\mathcal{W}(x,t) = \mathrm{Re}\{w(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$, the time-harmonic scattered wave is given by $\mathcal{V}(x,t) = \mathrm{Re}\{v(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$ where v is solution to

$$\Delta v + \omega^2\, v = 0 \quad \text{in } \mathbb{R}^2 \setminus \bigcup_{p=1}^{P} \mathcal{O}_p, \tag{3}$$

$$v = -w \quad \text{on } \bigcup_{p=1}^{P} \partial\mathcal{O}_p, \tag{4}$$

and satisfies in addition the usual Sommerfeld radiation condition

$$\frac{\partial v}{\partial |x|} - \mathrm{i}\omega\, v = O(|x|^{-3/2}) \quad \text{as } |x| \to \infty. \tag{5}$$

2. Asymptotics for small scatterers

Instead of the above system of equations, we consider a family of asymptotic models which are valid for small scatterers, more precisely when the diameters of the scatterers are small compared to the wavelength $2\pi/\omega$. These models are based on the fact that in the case of one scatterer $(P = 1)$, the solution v to (3)–(5) can be approximated by

$$\sigma_1 w(s_1) G(x - s_1),$$

where

$$G(x) := \frac{H_0^{(1)}(\omega\,|x|)}{4i} \tag{6}$$

is the outgoing Green's function of the Helmholtz equation ($H_0^{(1)}$ is the Hankel function of the first kind of order 0) and σ_1 is the reflection coefficient on the scatterer, which is given by $\sigma_1 = -4i/H_0^{(1)}(\omega r_1)$ for a circular obstacle. If $P > 1$, we can consider different levels of approximation $v^{(0)}, v^{(1)}, \ldots, v^{(\infty)}$ of v which consist in superpositions of the form

$$v^{(k)}(x) := \sum_{p=1}^{P} \sigma_p\, w_p^{(k)}\, G(x - s_p),$$

where $w_p^{(k)}$ represents different approximations of an "exciting field" on the p-th scatterer. In the simplest model $(k = 0)$, we choose $w_p^{(0)} = w(s_p)$, which amounts to neglecting the interactions between the obstacles. The case $k = \infty$ corresponds to the Foldy-Lax model [3], which takes into account these interactions. In this case, the exciting field is the superposition of the incident field and the waves scattered by all the other obstacles, i.e.,

$$w_p^{(\infty)} = w(s_p) + \sum_{q \neq p} \sigma_q w_q^{(\infty)} G(s_p - s_q) \quad \text{for } p = 1, \ldots, P. \tag{7}$$

If we denote by $W^{(\infty)}$ and W the vectors of $\mathbb{C}^P$ with components $w_p^{(\infty)}$ and $w(s_p)$ respectively, this coupling between the exciting fields can be written equivalently as the following linear system:

$$(\mathbb{I} + \mathbb{M})\, W^{(\infty)} = W, \tag{8}$$

where $\mathbb{M}$ is the $P \times P$ matrix defined by $\mathbb{M}_{pq} = -\sigma_q G(s_p - s_q)$ if $q \neq p$, and $\mathbb{M}_{pp} = 0$.

Between the cases $k = 0$ and $k = \infty$, one can consider intermediate models which take into account the successive reflections between the scatterers. Instead of (7), the exciting field is defined recursively by

$$w_p^{(k+1)} = w(s_p) + \sum_{q \neq p} \sigma_q w_q^{(k)} G(s_p - s_q) \quad \text{for } p = 1, \ldots, P.$$

It is readily seen that this relation amounts to approximating the inverse of operator $\mathbb{I} + \mathbb{M}$ involved in (8) by a truncated Neumann series, so that we can summarize these different models by the formula

$$W^{(k)} = \sum_{\ell=0}^{k} (-\mathbb{M})^\ell \, W \quad \text{for } k = 0, 1, \ldots, \infty. \tag{9}$$

It can be shown [2] that the error (in a local L^2 norm) is of order $|\log \varepsilon|^{-(k+2)}$ for finite k, and $\varepsilon / |\log \varepsilon|$ for $k = \infty$, where ε denotes the ratio of the greatest radius by the wavelength.

3. The time reversal operator

As described in the introduction, the TRO corresponds to two successive iterations of the following loop. In a first step, the TRM emits an incident time-harmonic wave given by

$$w(x) = \sum_{n=1}^{N} f_n \, G(x - x_n) \tag{10}$$

where $f_1, \ldots, f_N$ denote the complex amplitudes of the input signals at the N transducers $x_1, \ldots, x_N$. This wave interacts with the scatterers, and the TRM then measures the scattered wave. If we use for instance the Foldy–Lax model ($k = \infty$), the measure at the transducer x_n is

$$\begin{aligned} v^{(\infty)}(x_n) &= \sum_{p=1}^{P} \sigma_p \, w_p^{(\infty)} \, G(x_n - s_p) \\ &= \sum_{p=1}^{P} \sigma_p \, ((\mathbb{I} + \mathbb{M})^{-1} W)_p \, G(x_n - s_p) \\ &= \sum_{p=1}^{P} (\mathbb{D} \, (\mathbb{I} + \mathbb{M})^{-1} \mathbb{G} \, f)_p \, \mathbb{G}_{np}^\top \\ &= (\mathbb{G}^\top \, \mathbb{D} \, (\mathbb{I} + \mathbb{M})^{-1} \, \mathbb{G} \, f)_n, \end{aligned}$$

where $\mathbb{G}$ is the $P \times N$ matrix defined by $\mathbb{G}_{pn} = G(x_n - s_p)$, $\mathbb{D}$ is the $P \times P$ diagonal matrix defined by $\mathbb{D}_{pp} = \sigma_p$, and f is the vector of $\mathbb{C}^N$ with components f_n. We can then define the operator $\mathbb{F} \in \mathcal{L}(\mathbb{C}^N)$ which maps the input f to the measure of the scattered wave:

$$\mathbb{F} \, f := \mathbb{G}^\top \, \mathbb{D} \, (\mathbb{I} + \mathbb{M})^{-1} \, \mathbb{G} \, f. \tag{11}$$

The last step of the loop is to time-reverse the measure $\mathbb{F}\,f$, which is a simple complex conjugation in the frequency domain: the components of $\overline{\mathbb{F}\,f}$ can then be used as input signals to re-emit a new incident wave. Finally, two successive loops are represented by the following operator

$$\mathbb{T}\,f := \overline{\mathbb{F}\overline{\mathbb{F}\,f}} = \overline{\mathbb{F}}\,\mathbb{F}\,f = \mathbb{F}^{*}\mathbb{F}\,f, \tag{12}$$

where the last equality follows from the fact that $\mathbb{F}^{*} = \overline{\mathbb{F}}$ (which is easily deduced from (11)).

Note that we could define similarly a TRO associated with one of the approximate model which takes into account the k first reflections between the scatterers. In view of (9), we simply have to replace $(\mathbb{I}+\mathbb{M})^{-1}$ by $\sum_{\ell=0}^{k}(-\mathbb{M})^{\ell}$ in the definition (11) of $\mathbb{F}$.

It follows from (12) that $\mathbb{T}$ is a positive selfadjoint operator. Hence it can be diagonalized in an orthonormal basis of eigenvectors. The eigenelements of $\mathbb{T}$ have the following remarkable properties (which hold except in very particular symmetrical cases). First, the number of scatterers is equal to the number of nonzero eigenvalues of the TRO. Moreover, when these eigenvalues are simple, each eigenvector associated with one of them generates a wave which focuses selectively on each scatterer (see [4, 5]).

4. Space–time focusing

In the previous section, we have shown how to construct the time-reversal operator for a fixed frequency ω. Suppose that in a given frequency band $[\omega_1,\omega_2]$, we know an eigenvector $f(\omega)\in\mathbb{C}^N$ of the TRO, chosen such that $\|f(\omega)\|_{\mathbb{C}^N}=1$, which is associated with a given scatterer s_p, in the sense that the corresponding time-harmonic incident wave (10) focuses on this scatterer. For a function $A:\ [\omega_1,\omega_2]\to\mathbb{C}$, we can consider the superposition of the time-harmonic input signals given by

$$\mathcal{F}(t) = \mathrm{Re}\int_{\omega_1}^{\omega_2} A(\omega)\,f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{13}$$

which will generate the following time-dependent incident wave:

$$\mathcal{W}(x,t) = \mathrm{Re}\int_{\omega_1}^{\omega_2} A(\omega)\,\sum_{n=1}^{N} f_n(\omega)\,G(x-x_n;\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{14}$$

where we now indicate the dependence with respect to ω in the time-harmonic Green's function defined in (6). This wave focuses in space near s_p, since it is a superposition of focused waves. But how can we choose $A(\omega)$ so that it focuses also in time, that is, so that the period of interaction of $\mathcal{W}(x,t)$ with the scatterer is as short as possible? What kind of criterion can be used?

The idea we follow here is based on the fact that the best space–time focusing is obtained for the time-reversed Green's function of the wave equation $\mathcal{G}(x-s_p,-t)$ where

$$\mathcal{G}(x,t) = \frac{-H(t-|x|)}{2\pi(t^2-|x|^2)^{\frac{1}{2}}},$$

(H denotes the heaviside function). This function is related to the time-harmonic Green's function by the formula

$$\mathcal{G}(x,t) = \frac{1}{\pi}\,\mathrm{Re}\int_{0}^{+\infty} G(x;\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

As a consequence, the measures at the transducers of $\mathcal{G}(x - s_p, t)$ are given by

$$\begin{pmatrix} \mathcal{G}(x_1 - s_p, t) \\ \vdots \\ \mathcal{G}(x_N - s_p, t) \end{pmatrix} = \frac{1}{\pi} \operatorname{Re} \int_0^{+\infty} \Gamma_p(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega$$

where $\Gamma_p(\omega) := (G(x_1 - s_p; \omega), \ldots, G(x_N - s_p; \omega))^\top$. The time-reversed measures are thus obtained by replacing $\Gamma_p(\omega)$ by its conjugate. In order to obtain a signal of the expected form (13), it is then natural to replace $\overline{\Gamma_p(\omega)}$ by its orthogonal projection on the eigenspace spanned by $f(\omega)$, which is given by $(\overline{\Gamma_p(\omega)}, f(\omega))_{\mathbb{C}^n}\, f(\omega)$. Using a cutoff function $\chi :\ \mathbb{R}^+ \to \mathbb{R}^+$ with support in the imposed frequency band $[\omega_1, \omega_2]$, the expected function $A(\omega)$ in (13) has the form

$$A(\omega) = \chi(\omega)\ (\overline{\Gamma_p(\omega)}, f(\omega))_{\mathbb{C}^n} \quad \forall \omega \in [\omega_1, \omega_2].$$

We shall present some numerical results which show that the incident wave (14) corresponding to this choice of $A(\omega)$ actually focuses in space and time near s_p. We shall also consider the more involved situation where the propagative medium contains a diffusive region modeled by a random distribution of pointlike scatterers, which improves the focusing effect.

Let us notice that this is only a numerical confirmation of the focusing effect. Some related mathematical questions remain open. On one hand, is there a mathematical definition of focusing that could allow us to evaluate the quality of a focusing wave? On the other hand, can we find a criterion that leads us to the above choice of $A(\omega)$ or even a better one? Indeed, in the above lines, the position of the scatterer s_p is known *a priori*, since we have assumed that $\Gamma_p(\omega)$ is known. But can we do without this knowledge, using only the measures of the TRM? Works on these issues are in progress.

5. References

[1] C. BEN AMAR, C. HAZARD, "Time reversal and scattering theory for time-dependent acoustic waves in a homogeneous medium", *IMA J. Appl. Math.*, to appear.

[2] M. CASSIER, C. HAZARD, "Multiple acoustic scattering by small obstacles in two dimensions", in preparation.

[3] L.L. FOLDY, "The multiple Scattering of Waves, I. General Theory of Isotropic Scattering by Randomly Distribued Scatterers", *Physical Review*, vol. 67, 107–119, 1945.

[4] C. HAZARD, K. RAMDANI, "Selective acoustic focusing using time-harmonic reversal mirrors", *SIAM J. Appl. Math.*, vol. 64, 1057–1076, 2004.

[5] C. PRADA, S. MANNEVILLE, D. SPOLIANSKY, M. FINK, "Decomposition of the time reversal operator: Detection and selective focusing on two scatterers", *J. Acoust. Soc. Am.*, vol. 99, 2067–2076, 1996.