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# The linear sampling method in a waveguide : a formulation based on modes

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**Abstract.** This paper concerns the Linear Sampling Method to retrieve obstacles in a 2D or 3D acoustic waveguide. We derive a modal formulation of the LSM which is suitable for the waveguide configuration. Despite the ill-posedness of the inverse problem is increased owing to the evanescent modes, numerical experiments show good reconstruction of obstacles by using the far field.

## 1. Introduction

The Linear Sampling Method, first introduced by D. Colton and A. Kirsch [1], has been extensively used for solving inverse diffraction problems in acoustics, as can be seen in [2]. The main attractive feature of such method is that it does not require *a priori* knowledge about the boundary condition on the boundary of the unknown scatterer. However, it seems to the authors that the Linear Sampling Method in a waveguide has been much less treated, despite some articles have connections with this subject [3, 4, 5, 6]. It is well known that imaging a scatterer in a waveguide is much more challenging than in free space. Indeed, because of the presence of the boundary of the waveguide, only a finite number of modes can propagate at long distance, while the other modes decay exponentially as a function of distance.

The following paper concerns the Linear Sampling Method in a 2D or 3D acoustic waveguide and proposes a formulation of the method that uses modes. Essentially, we are interested in the far field situation, which means that the support of data is a section of the waveguide which is far away from the obstacle. From a practical point of view, for example in the field of non destructive evaluation, this situation is of interest.

The paper is organized as follows. In section 2, we recall some basic properties of the forward problem. In section 3, we introduce the inverse problem and describe our modal formulation of the LSM. Lastly, numerical results in 2D are presented in section 4. The principles of the method are described briefly in this paper, but we refer to [7] for all mathematical justifications.

## 2. The forward problem

We consider a waveguide of domain $W = \mathbb{R} \times \Sigma$ in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. In 2D, $\Sigma = (0, h)$, where $h > 0$, while in 3D, $\Sigma$ is a bounded and open $C^\infty$-class domain of $\mathbb{R}^2$, the boundary of which is denoted $\Gamma$. In the following, $x = (x_1, x_2)$ will denote a generic point of $W$, where $x_1 \in \mathbb{R}$ and $x_2 \in \Sigma$.

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Classically, the solutions of the Neumann eigenvalue problem for the negative Laplacien in $\Sigma$ form an increasing sequence $k_n^2 \in \mathbb{R}^+$ for $n \in \mathbb{N}$, with $k_0 = 0$ and $k_n \to +\infty$ when $n \to +\infty$, and we can find some eigenvectors $\theta_n$ which form an orthonormal basis of $L^2(\Sigma)$.
The solutions of the problem

$$\left\{ \begin{array}{ccc} (\Delta + k^2)u = 0 & in & W \\ \partial_\nu u = 0 & on & \Gamma, \end{array} \right. \tag{1}$$

where $\nu$ is the outward unit normal on $\Gamma$, are the linear combinations of the guided modes, defined for $n \in \mathbb{N}$ by

$$g_n^\pm(x_1, x_2) = \theta_n(x_2) e^{\pm i\beta_n x_1}, \quad \beta_n = \sqrt{k^2 - k_n^2}, \quad \operatorname{Re}\beta_n, \operatorname{Im}\beta_n \geq 0, \tag{2}$$

and we assume that $\beta_n \neq 0$ for all $n \geq 0$. There is a finite number $N_p$ such that the $N_p$ first $\beta_n$ have $\operatorname{Re}\beta_n > 0$, the rest of the $\beta_n$ having $\operatorname{Im}\beta_n > 0$. Consequently, the $N_p$ first guided modes $g_n^+$ (respectively $g_n^-$) are propagating from the left to the right of the waveguide (respectively from the right to the left), while the other guided modes $g_n^+$ (respectively $g_n^-$) are decaying exponentially from the left to the right of the waveguide (respectively from the right to the left).
We define now $\Sigma_s = \{s\} \times \Sigma$. Let $\mathcal{D}$ be an open bounded domain within $W$, the boundary of which is Lipschitz continuous and denoted $\partial D$. This obstacle is assumed to lie between sections $\Sigma_{-t_0}$ and $\Sigma_{t_0}$, with $t_0 > 0$.
For $k > 0$ and $f \in H^{1/2}(\partial\mathcal{D})$, the forward problem we consider is

$$\left\{ \begin{array}{ccc} (\Delta + k^2)u = 0 & in & W \setminus \overline{\mathcal{D}} \\ \partial_\nu u = 0 & on & \Gamma \\ u = f & on & \partial\mathcal{D} \\ \partial_\nu u = T^\pm u & on & \Sigma_{\pm t}. \end{array} \right. \tag{3}$$

Here, $t > t_0$ and $T^\pm$ is the Dirichlet to Neumann operator $T^\pm : H^{1/2}(\Sigma_{\pm t}) \longrightarrow H^{-1/2}(\Sigma_{\pm t})$, defined for $h \in H^{1/2}(\Sigma_{\pm t})$ by

$$T^\pm h = \sum_{n \in \mathbb{N}} i\beta_n (h, \theta_n)_{\Sigma_{\pm t}} \theta_n,$$

where $(.,.)_{\Sigma_s}$ is the standard scalar product in $L^2(\Sigma_s)$.
The solution $u$ of problem (3) is the scattered field due to the soft obstacle $\mathcal{D}$, where the data $-f$ is the trace on $\partial\mathcal{D}$ of an incident wave. The radiation condition consists of the last condition in problem (3). It means that the scattered field $u$ is a superposition of guided modes that are outgoing. It is well-known that for a given obstacle $\mathcal{D}$ the problem (3) is well-posed in $H^1_{loc}(W \setminus \overline{\mathcal{D}})$, except for at most a countable set of wavenumbers $k$. Throughout this paper, we assume that $k$ is such that problem (3) is well-posed.

In the following, we need to introduce the Green function of the waveguide $W$, denoted $G(x, y)$.
For a given point $y = (y_1, y_2) \in W$, $G(., y)$ is the solution of problem

$$\left\{ \begin{array}{ccc} (\Delta_x + k^2)G(., y) = \delta_y & in & W \\ \partial_{\nu_x} G(., y) = 0 & on & \Gamma \\ \partial_{\nu_x} G(., y) = T^\pm G(., y) & on & \Sigma_{\pm t}. \end{array} \right.$$

The function $G$ is given, for all $x, y \in W$, by

$$G(x, y) = \sum_{n \in \mathbb{N}} \frac{e^{i\beta_n |x_1 - y_1|}}{2i\beta_n} \theta_n(x_2)\theta_n(y_2). \tag{4}$$

We note that $G(., y) \notin H^1_{loc}(W)$ both in 2D and 3D. Let us introduce the solution $u^s(., y)$ of the problem (3) with $f = -G(., y)|_{\partial\mathcal{D}}$, $y \notin \partial\mathcal{D}$.

## 3. The inverse problem and the Linear Sampling Method

We assume that we impose an incident wave of form $G(.,y)$ for $y \in \hat{\Sigma}$ and we measure the corresponding scattered field $u^s(x,y)$ due to the obstacle $\mathcal{D}$ for $x \in \hat{\Sigma}$. For $\hat{\Sigma}$, we consider either of the two definitions :

$$(1) \quad \hat{\Sigma} = \Sigma_{-s} \cup \Sigma_s \quad (2) \quad \hat{\Sigma} = \Sigma_s$$

for some $s > t_0$. The case (1) will be called the full aperture situation, while the case (2) will be called the back-scattering situation. In the two cases, the objective is to determine $\mathcal{D}$ from the measurements of $u^s(.,y)$ on $\hat{\Sigma}$ for all $y \in \hat{\Sigma}$, under the following assumption.

***Assumption*** $(H)$ *:* $k^2$ *is not a Dirichlet eigenvalue for the negative Laplacian in* $\mathcal{D}$.

Secondly, we introduce the same type of operators as in [8] with similar notations. Let us define the near field operator $F \ : L^2(\hat{\Sigma}) \to L^2(\hat{\Sigma})$ such that for $h \in L^2(\hat{\Sigma})$,

$$(Fh)(x) = \int_{\hat{\Sigma}} u^s(x,y)h(y)\,ds(y), \quad x \in \hat{\Sigma}, \tag{5}$$

the operator $B \ : H^{1/2}(\partial\mathcal{D}) \to L^2(\hat{\Sigma})$ such that for $f \in H^{1/2}(\partial\mathcal{D})$, $Bf = u|_{\hat{\Sigma}}$, where $u$ is the solution of problem (3). Next, we define $\mathcal{H} : L^2(\hat{\Sigma}) \to H^{1/2}(\partial\mathcal{D})$ such that for $h \in L^2(\hat{\Sigma})$, $\mathcal{H}h = v_h|_{\partial\mathcal{D}}$, where $v_h$ is the function defined by

$$v_h(x) = \int_{\hat{\Sigma}} G(x,y)h(y)\,ds(y). \tag{6}$$

Lastly, we define $\mathcal{F} : H^{-1/2}(\partial\mathcal{D}) \to L^2(\hat{\Sigma})$ and $S : H^{-1/2}(\partial\mathcal{D}) \to H^{1/2}(\partial\mathcal{D})$ by $(\mathcal{F}\phi)(x) = w_\phi|_{\hat{\Sigma}}$ and $(S\phi)(x) = w_\phi|_{\partial\mathcal{D}}$, where $w_\phi$ is the function defined by

$$w_\phi(x) = \int_{\partial\mathcal{D}} \phi(y)G(x,y)\,ds(y).$$

The operators $S$, $\mathcal{H}$, $\mathcal{F}$ and $F$ satisfy the following properties.

***Proposition 1*** *If assumption* $(H)$ *holds,* $S$ *is an isomorphism,* $\mathcal{H}$*,* $\mathcal{F}$ *and* $F$ *are compact operators, injective with dense range.*

As in [8], by superposition we straightforward obtain the relationships

$$F = -B\mathcal{H}, \quad \mathcal{F} = BS, \tag{7}$$

and then we obtain the following factorization of $F$:

$$F = -\mathcal{F}S^{-1}\mathcal{H}. \tag{8}$$

We first verify that the measurements of $u^s(.,y)$ on $\hat{\Sigma}$ for all $y \in \hat{\Sigma}$ uniquely determine the soft obstacle $\mathcal{D}$. The following theorem is obtained by using the same proof as in [9] (see also [10]), and relies on lemmas 1, 2 and 3 of [7]. Lemma 3 consists of the reciprocity relationship:

$$\forall x, y \in W \setminus \overline{\mathcal{D}}, \quad u^s(x,y) = u^s(y,x).$$

.

***Theorem 1*** *Let us denote by* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *two soft obstacles whose boundaries are Lipschitz continuous. If we assume that for all incident waves* $G(.,y)$ *with* $y \in \hat{\Sigma}$*, the corresponding scattered fields* $u_1^s(.,y)$ *and* $u_2^s(.,y)$ *coincide on* $\hat{\Sigma}$*, then* $\mathcal{D}_1 = \mathcal{D}_2$.

The Linear Sampling Method is based on the following equation : find $h(.,z) \in L^2(\hat{\Sigma})$ such that

$$Fh = G(.,z), \tag{9}$$

where $z$ is a sampling point of $W$. The method consists in finding a quasi-solution of (9) for each $z$ and then in plotting the norm of $h(.,z)$, which happens to explode outside the obstacle $\mathcal{D}$. This method is justified in part by the following theorem, which is very similar to the one presented in [8].

***Theorem 2*** *We assume that assumption (H) is satisfied for an obstacle $\mathcal{D}$ with Lipschitz continuous boundary. Let $F$ be the near field operator defined by (5) with $u^s(.,y)$ being the solution of problem (3) with $f = -G(.,y)|_{\partial\mathcal{D}}$.*
(1) *If $z \in \mathcal{D}$, then for all $\varepsilon > 0$ there exists a solution $h_\varepsilon(.,z) \in L^2(\hat{\Sigma})$ of the inequality*

$$||Fh_\varepsilon(.,z) - G(.,z)||_{L^2(\hat{\Sigma})} \leq \varepsilon$$

*such that the function $\mathcal{H}h_\varepsilon(.,z)$ converges in $H^{1/2}(\partial D)$ as $\varepsilon \to 0$.*
*Furthermore, for a given fixed $\varepsilon$, the function $h_\varepsilon(.,z)$ satisfies*

$$\lim_{z\to\partial\mathcal{D}} ||h_\varepsilon(.,z)||_{L^2(\hat{\Sigma})} = \infty, \quad \lim_{z\to\partial\mathcal{D}} ||\mathcal{H}h_\varepsilon(.,z)||_{H^{1/2}(\partial D)} = \infty.$$

(2) *If $z \in W \setminus \overline{\mathcal{D}}$, then every solution $h_\varepsilon(.,z) \in L^2(\hat{\Sigma})$ of the inequality*

$$||Fh_\varepsilon(.,z) - G(.,z)||_{L^2(\hat{\Sigma})} \leq \varepsilon$$

*satisfies*

$$\lim_{\varepsilon\to 0} ||h_\varepsilon(.,z)||_{L^2(\hat{\Sigma})} = \infty, \quad \lim_{\varepsilon\to 0} ||\mathcal{H}h_\varepsilon(.,z)||_{H^{1/2}(\partial D)} = \infty.$$

**Proof** : (1) we first consider the case $z \in \mathcal{D}$, for which clearly $-G(.,z)|_{L^2(\hat{\Sigma})} \in \mathrm{Im}B$, that is thank's to (7), $-G(.,z)|_{L^2(\hat{\Sigma})} \in \mathrm{Im}\mathcal{F}$. Since $\mathcal{F}$ is injective,

$$\exists!\, \phi_z \in H^{-1/2}(\partial\mathcal{D}), \quad \mathcal{F}\phi_z = G(.,z)|_{L^2(\hat{\Sigma})}.$$

On the other hand, $-S\phi_z \in H^{1/2}(\partial\mathcal{D})$ and $\mathcal{H}$ has dense range, thus for all $\varepsilon > 0$ there exists $h_{\varepsilon,z} \in L^2(\hat{\Sigma})$ such that

$$||\mathcal{H}h_{\varepsilon,z} + S\phi_z||_{H^{1/2}(\partial\mathcal{D})} \leq \varepsilon,$$

and since $\mathcal{F}S^{-1}$ is continuous, for some $c > 0$

$$||\mathcal{F}S^{-1}\mathcal{H}h_{\varepsilon,z} + \mathcal{F}\phi_z||_{L^2(\hat{\Sigma})} = ||Fh_{\varepsilon,z} - G(.,z)||_{L^2(\hat{\Sigma})} \leq c\,\varepsilon,$$

which is, together with $\mathcal{H}h_{\varepsilon,z} \to -S\phi_z$ in $H^{1/2}(\partial\mathcal{D})$ as $\varepsilon \to 0$, the claimed result.
Moreover, we have that $G(.,z) \notin H^1_{loc}(W)$ for $z \in W \setminus \mathcal{D}$, so that for fixed $\varepsilon$, when $z \to \partial\mathcal{D}$ with $z \in \mathcal{D}$,

$$||G(.,z)||_{H^{1/2}(\partial\mathcal{D})} = ||S\phi_z||_{H^{1/2}(\partial\mathcal{D})} \to \infty,$$

which implies $||\mathcal{H}h_{\varepsilon,z}||_{H^{1/2}(\partial\mathcal{D})} \to \infty$, and finally $||h_z||_{L^2(\hat{\Sigma})} \to \infty$ since $\mathcal{H}$ is continuous.

(2) we secondly consider the case $z \notin \mathcal{D}$, for which $-G(.,z)|_{L^2(\hat{\Sigma})} \notin \mathrm{Im}B = \mathrm{Im}\mathcal{F}$. For each $h_{\varepsilon,z} \in L^2(\hat{\Sigma})$ such that $||Fh_{\varepsilon,z} - G(.,z)||_{L^2(\hat{\Sigma})} \leq \varepsilon$ (such a function exists since $F$ has dense

range), we define $\phi_{\varepsilon,z} \in H^{-1/2}(\partial\mathcal{D})$ by $\phi_{\varepsilon,z} = S^{-1}\mathcal{H}h_{\varepsilon,z}$, whence $-Fh_{\varepsilon,z} = \mathcal{F}\phi_{\varepsilon,z}$, and as a result

$$||\mathcal{F}\phi_{\varepsilon,z} + G(.,z)||_{L^2(\hat{\Sigma})} \leq \varepsilon.$$

For a fixed $z$, $\lim_{\varepsilon\to 0} ||\phi_{\varepsilon,z}||_{H^{-1/2}(\partial D)} = +\infty$. If not, we could find a subsequence still denoted $\phi_{\varepsilon,z}$ that weakly converges to $\phi_z$ in $H^{-1/2}(\partial D)$, and since $\mathcal{F}$ is compact, $\mathcal{F}\phi_{\varepsilon,z}$ strongly converges to $\mathcal{F}\phi_z$ in $L^2(\hat{\Sigma})$. This would lead to $\mathcal{F}\phi_z = -G(.,z)|_{\hat{\Sigma}}$ by using the above inequality, which is not possible. As a result, by using continuity of $S^{-1}$,

$$\lim_{\varepsilon\to 0} ||S\phi_{\varepsilon,z}||_{H^{1/2}(\partial\mathcal{D})} = \infty,$$

and finally,

$$\lim_{\varepsilon\to 0} ||\mathcal{H}h_{\varepsilon,z}||_{H^{1/2}(\partial\mathcal{D})} = \infty, \quad \lim_{\varepsilon\to 0} ||h_{\varepsilon,z}||_{L^2(\hat{\Sigma})} = \infty,$$

which completes the proof. ■

We now present a modal version of the Linear Sampling Method, which is obtained by projecting the relationship (9) upon the functions $\theta_n$, which form an orthonormal basis in $L^2(\Sigma)$. This is motivated by the particular form of the Green function of the waveguide (4). We denote by $u_n^{\pm}$ the solution for $n \in \mathbb{N}$ of problem (3) with $f = -g_n^{\pm}|_{\partial\mathcal{D}}$. We assume that $h = (h^-, h^+) \in L^2(\Sigma_{-s}) \times L^2(\Sigma_s)$, the decomposition of $h^-$ (resp. $h^+$) in terms of the $\theta_n$ being denoted $h_n^-$ (resp. $h_n^+$).
Let us denote for $x \in \Sigma_{\pm s}$,

$$u_n^+(x) = \sum_{m\in\mathbb{N}} (U_n^+)_m^{\pm}\, \theta_m(x_2), \quad u_n^-(x) = \sum_{m\in\mathbb{N}} (U_n^-)_m^{\pm}\, \theta_m(x_2).$$

In [7], it is proven that the equation (9) in the full aperture situation is equivalent to

$$\forall m \in \mathbb{N}, \quad \left\{ \begin{array}{l} \displaystyle\sum_{n\in\mathbb{N}} \frac{e^{i\beta_n s}}{i\beta_n} \left((U_n^+)_m^-\, h_n^- + (U_n^-)_m^-\, h_n^+\right) = \frac{e^{i\beta_m(s+z_1)}}{i\beta_m}\theta_m(z_2) \\ \displaystyle\sum_{n\in\mathbb{N}} \frac{e^{i\beta_n s}}{i\beta_n} \left((U_n^+)_m^+\, h_n^- + (U_n^-)_m^+\, h_n^+\right) = \frac{e^{i\beta_m(s-z_1)}}{i\beta_m}\theta_m(z_2), \end{array} \right. \tag{10}$$

while in the back-scattering situation (9) is equivalent to

$$\forall m \in \mathbb{N}, \quad \sum_{n\in\mathbb{N}} \frac{e^{i\beta_n s}}{i\beta_n} (U_n^-)_m^+\, h_n^+ = \frac{e^{i\beta_m(s-z_1)}}{i\beta_m}\theta_m(z_2). \tag{11}$$

One of the more important fact concerning the Linear Sampling Method in a waveguide is that, as the equations (10) and (11) show it :

(i) in the full aperture situation, measuring for $x \in \Sigma_{\pm s}$ the scattered fields $u^s(x,y)$ due to the incident waves of form $G(.,y)$ for $y \in \Sigma_{\pm s}$ is equivalent to measuring the projections on the $\theta_m$ of the scattered fields $u_n^{\pm}(x)$ for $x \in \Sigma_{\pm s}$, due to the incident waves $g_n^{\pm}$ for all $m, n \in \mathbb{N}$,

(ii) in the back-scattering situation, measuring for $x \in \Sigma_s$ the scattered fields $u^s(x,y)$ due to the incident waves of form $G(.,y)$ for $y \in \Sigma_s$ is equivalent to measuring the projections on the $\theta_m$ of the scattered fields $u_n^-(x)$ for $x \in \Sigma_s$, due to the incident waves $g_n^-$ for all $m, n \in \mathbb{N}$.

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For numerical purpose, we restrict ourselves to a finite number $M$ of indices $m$ and a finite number $N$ of indices $n$ in equations (10) and (11), since we impose a finite number of incident modes, and we project the measurements of the corresponding scattered fields upon a finite number of transverse eigenfunctions. For simplicity, we assume that $M = N$, and that these $N$ modes correspond to $N_p$ propagating modes and $N_e$ evanescent modes. Therefore, the equations (10) and (11) lead to a system

$$UH = C. \tag{12}$$

For example, in the back-scattering situation, the $N \times N$ matrix $U$ and the $N$ vectors $H, C$ are defined respectively by

$$U_{mn} = \frac{e^{i\beta_n s}}{i\beta_n}(U_n^-)_m^+, \quad H_n = h_n^+, \quad C_m = \frac{e^{i\beta_m(s-z_1)}}{i\beta_m}\theta_m(z_2). \tag{13}$$

If we separate propagating and evanescent modes, (12) can be written as follows :

$$\begin{pmatrix} U^{pp} & U^{pe} \\ U^{ep} & U^{ee} \end{pmatrix} \begin{pmatrix} H^p \\ H^e \end{pmatrix} = \begin{pmatrix} C^p \\ C^e \end{pmatrix}. \tag{14}$$

This is now a natural question to wonder if we shall restrict ourselves to the propagating modes, that is we solve $U^{pp}H^p = C^p$ rather than (14), or if we shall also use the evanescent ones. For sake of simplicity, we try to answer this question in the back-scattering situation. First, we define $s_0 = \sup_{x \in \mathcal{D}} x_1$. Without loss of generality, in this section the origin of axis $x_1$ is chosen such that $s_0 = 0$. With this convention, $s$ is the distance between the obstacle and the section $\Sigma_s$ where back-scattering measurements take place. Given definition (13), we prove that for $0 \le m, n \le N-1$,

$$U_{mn} = e^{i\beta_m s} e^{i\beta_n s} U'_{mn}, \quad \text{with} \quad U'_{mn} := \frac{1}{i\beta_n}(u_n^-(0, x_2), \theta_m)_{\Sigma_0}.$$

We denote by $K$ the diagonal $N \times N$ matrix which is formed by the $e^{i\beta_m s}$, and we decompose $K$ into its propagating part $K^p$ and its evanescent part $K^e$. Similarly, given definition (13),

$$C_m = e^{i\beta_m(s-z_1)} C'_m, \quad \text{with} \quad C'_m = \frac{1}{i\beta_m}\theta_m(z_2),$$

and we denote by $L = (L^p, L^e)$ the diagonal $N \times N$ matrix which is formed by the $e^{i\beta_m(s-z_1)}$. Remembering that in the matrices $K^p$ and $L^p$, $\beta_m = \sqrt{k^2 - k_m^2} \in \mathbb{R}^+$ and that in the matrices $K^e$ and $L^e$, $\beta_m := i\gamma_m$ with $\gamma_m = \sqrt{k_m^2 - k^2} \in \mathbb{R}^+$, for sufficiently large values of $s$ the matrices $K^e$ and $L^e$ are small matrices compared to matrices $K^p$ and $L^p$. Using these definitions, the system (14) reads by denoting $V^{pe} = K^p U'^{pe}$ and $V^{ep} = U'^{ep} K^p$,

$$\begin{pmatrix} U^{pp} & V^{pe}K^e \\ K^e V^{ep} & K^e U'^{ee} K^e \end{pmatrix} \begin{pmatrix} H^p \\ H^e \end{pmatrix} = \begin{pmatrix} C^p \\ L^e C'^e \end{pmatrix}. \tag{15}$$

Thus, it is clear that the blocks of the system in which $K^e$ and $L^e$ appear almost vanish for high values of $s$, and hence the system (15) becomes strongly ill-conditioned. Therefore, unless the distance $s$ between the obstacle and the support of data $\hat{\Sigma}$ is small, we will solve $U^{pp}H^p = C^p$ rather than $UH = C$, which means we will use only the propagating modes.

## 4. Some numerical results

In our numerical experiments, we consider a 2D waveguide of height $h = 1$, and we perform the modal version of the Linear Sampling Method we described in the previous section, both in the full aperture and the back-scattering situations. In presence of noisy data, we solve the Tikhonov/Morozov system associated to equation (12) as described in [7], for each right hand side $C(z)$ when $z$ takes all values of the sampling grid $[-t_0, t_0] \times [0, h]$, with $t_0 = \min(1, s)$. The figures hereafter represent the level curves of function $\log(1/||H(z)||)$, so that the retrieved obstacle is the set of points $z$ for which this function does not vanish.

We have tested two different obstacles :

(i) the obstacle is a sphere centered at $(0, 0.6)$ and of radius 0.2,

(ii) the obstacle consists of two spheres, the first one is centered at $(-0.2, 0.7)$ and is of radius 0.05, while the second one is centered at $(0.3, 0.5)$ and is of radius 0.07.

The synthetic data $u_n^{\pm}$ on $\hat{\Sigma} = \Sigma_{-s} \cup \Sigma_s$ (resp. $= \Sigma_s$) in the full aperture situation (resp. back-scattering situation) are obtained by using a classical finite element method based on a weak formulation of problem (3). The Dirichlet-to-neumann condition on the artificial sections $\Sigma_{\pm t}$ requires the projection of the finite element solution upon the transverse eigenfunctions $\theta_n$, which diagonalize the Dirichlet-to-neumann operator.

In the following we study how the quality of the reconstruction depends on the frequency and on the amplitude of noise. In [7], the importance of taking the evanescent modes into account is also analyzed. In the case $s = 2$, which is clearly a far field situation, we present the results obtained for the first obstacle $(i)$ for $k = 10$ and $k = 30$, the corresponding numbers of propagating modes being respectively $N_p = 4$ and $N_p = 10$. The figure (1) shows the results of the reconstruction both in the full aperture and the back-scattering situations. As expected, the quality is better

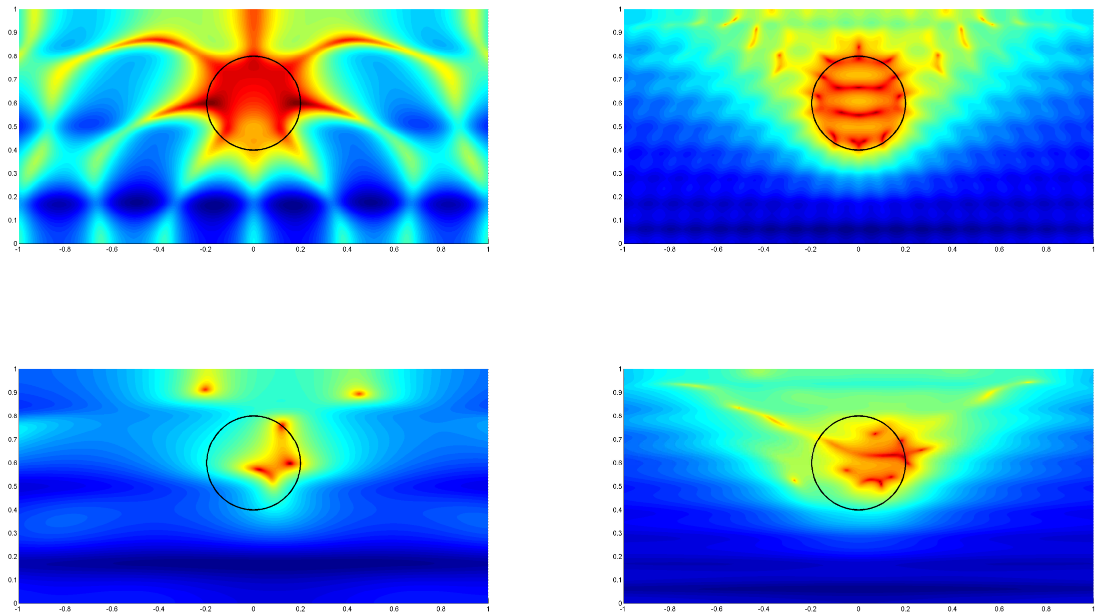

**Figure 1.** Full aperture : $k = 10$ ($N_p = 4$) (top left), $k = 30$ ($N_p = 10$) (top right). Back-scattering : $k = 10$ (bottom left), $k = 30$ (bottom right)

when we increase the number of propagating incident modes $N_p$, or equivalently when we increase the wavenumber $k$. Besides, reconstruction is logically better in the full aperture case than in the back-scattering case, particularly in the area of the obstacle which is not enlightened in the latter case.

The data $u_n^{\pm}$ obtained with our forward finite element computations are now subjected pointwise to a proportional Gaussian noise of amplitude 20%. We present hereafter the impact of the amplitude of noise on the quality of the reconstruction of the second obstacle $(ii)$, with $k = 30$. The figure (2) shows the results of the reconstruction both in the full aperture and the back-scattering situations, with or without noise. We notice that the quality of the reconstruction is not much affected by the noise which contaminates the data.

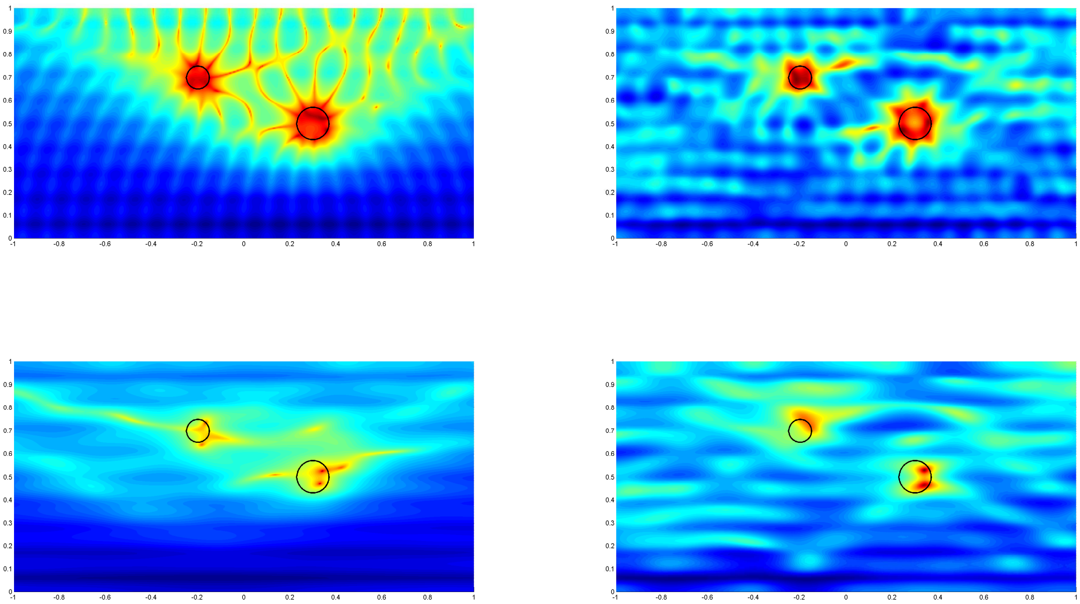

**Figure 2.** Full aperture : no noise (top left), 20% noise (top right). Back-scattering : no noise (bottom left), 20% noise (bottom right)

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