# A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation

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**Abstract.** This work concerns the use of the method of quasi-reversibility to solve the Cauchy problem for Laplace's equation. We describe a mixed formulation of that method and its relationship with a classical formulation. A discretized formulation using finite elements of class $C^0$ is derived from the mixed formulation, and convergence of the solution of that discretized problem with noisy data to the exact solution is analyzed. Finally, a simple numerical example is implemented in order to show the feasibility of the method.

## 1. Introduction

We consider the Cauchy problem for Laplace's equation in a bounded domain of $\mathbb{R}^N$ ($N = 2, 3$). It is well-known that such a problem is ill-posed in the sense of Hadamard (see [11] and the bibliography of [18]). It arises in many fields of physics, like electrocardiography (see [9]), plasma physics (see [1]), corrosion non-destructive evaluation (see [16] and [14]), or mechanical engineering (see [3]). Several methods of regularization can be used to solve the problem, some of those being presented in [3]. For example, the Cauchy equations (see [6]), which consist of transforming the initial partial derivative equation into a dynamical system, can be solved provided filtering methods are used. A general framework for that method is also presented in [23]. Otherwise, optimal control techniques (see [21], [1]), and approximate control techniques (see [2]), are natural methods to recover the lacking data on a part of the boundary. Other kinds of iterative algorithms recently emerged, like the one proposed in [19], which consists of solving a sequence of well-posed problems involving successively the Dirichlet data and the Neumann data, or the one presented in [8], which is based on a Tikhonov regularization, and which shows accuracy and robustness.

The method of quasi-reversibility was first proposed in [20] in the late 60's to solve the Cauchy problem for elliptic equations, which consists of transforming the ill-posed two-order initial problem into a family (depending on a small parameter $\varepsilon$) of fourth-order problems. In [18], the authors address the delicate question of the rate of convergence of the quasi-reversibility solution to the exact one, in particular with data errors, which was not considered in [20]. Their results are however strongly dependent on the geometry of the domain.

The quasi-reversibility method is an elegant non-iterative method which can easily be carried out numerically using the finite element method. The main drawback of such discretization stems from the smoothness of the finite elements used due to the order of the problem. These finite elements must be of class $C^1$ (see [7] for a description of such elements), which are rather cumbersome compared to usual finite elements of class $C^0$. Besides, seldom are these elements available in numerical codes. This observation probably explains why only finite difference schemes, and not finite element methods, have been up to now actually implemented for quasi-reversibility (to the author's knowledge). The aim of this paper is to propose a quasi-reversibility method that enables one to use classical finite elements, say of class $C^0$. It relies on mixed formulations, which have been widely used in many computational domains from the 70's (see [5] for a general description of mixed formulations).

The second section is devoted to the presentation of the Cauchy problem for Laplace's equation. The third section presents a classical fourth-order quasi-reversibility formulation which enables one to find an approximate solution to that problem. The fourth section considers a mixed formulation of quasi-reversibility and it exhibits the relationship it has with the classical formulation. Convergence of the solution of quasi-reversibility to the exact solution is analyzed, both with uncontaminated and with noisy data. The fifth section describes the discretized finite element formulation corresponding to the above mixed formulation, with particular attention to convergence of the discretized solution to the exact solution, with noisy data and on regularity assumptions. Finally, the sixth section presents a simple numerical example showing the feasibility of the mixed formulation of quasi-reversibility.

## 2. The Cauchy problem for Laplace's equation

Let $\Omega$ be a bounded, connected open set of $\mathbb{R}^N$ ($N = 2, 3$), of sufficiently "smooth" boundary $\Gamma = \partial\Omega$, i.e. either of class $C^1$ (see [12]) or polygonal ($N = 2$)/polyhedral ($N = 3$). Let $\Gamma_0$ be a "smooth" subset of $\Gamma$ (see [15]) with $\text{mes}(\Gamma_0) > 0$, and $\Gamma_1 = \Gamma/\Gamma_0$ with $\text{mes}(\Gamma_1) > 0$.

The Cauchy problem consists of finding $u \in H^1(\Omega)$ such that

$$\Delta u = 0 \text{ in } \Omega \tag{1}$$

and

$$\begin{cases} u|_{\Gamma_0} = g_0 \\ \dfrac{\partial u}{\partial n}|_{\Gamma_0} = g_1, \end{cases} \tag{2}$$

where $n$ is the normal vector outside $\Omega$, where $g_0 \in H^{\frac{1}{2}}(\Gamma_0)$ and $g_1 \in H^{-\frac{1}{2}}(\Gamma_0)$, $H^{-\frac{1}{2}}(\Gamma_0)$ being defined as the dual space of $H^{\frac{1}{2}}_{00}(\Gamma_0)$, which is itself defined by

$$H^{\frac{1}{2}}_{00}(\Gamma_0) = \{v \in L^2(\Gamma_0)\,;\, \exists w \in H^1(\Omega),\, w|_{\Gamma_0} = v,\, w|_{\Gamma_1} = 0\}.$$

$H^{\frac{1}{2}}_{00}(\Gamma_0)$ is hence a subspace of $H^{\frac{1}{2}}(\Gamma_0)$ (see [12] for details on these spaces).

We also define the Hilbert space $H^1(\Delta,\Omega) = \{u \in H^1(\Omega)\,;\, \Delta u \in L^2(\Omega)\}$ equipped with the norm $||.||_{H^1(\Delta,\Omega)}$ such that $||u||^2_{H^1(\Delta,\Omega)} = ||u||^2_{H^1(\Omega)} + ||\Delta u||^2_{L^2(\Omega)}$.

It is well known that this problem is ill-posed in the sense of Hadamard because the existence of a solution $u$ and its stability with respect to the data $g_0$ and $g_1$ don't hold even if these data are very smooth (see for example [11]).

However, we have the following uniqueness result.

***Lemma 1*** :
*There is at most one solution $u \in H^1(\Omega)$ which satisfies (1) and (2).*

***Proof*** :
This result is for example a consequence of the uniqueness theorem proved in [17] (see the appendix B, p. 75). This theorem states that if $u \in H^2(\Omega)$ satisfies the equation $\Delta u + b.\nabla u + au = 0$ in $\Omega$, where $a$ and $b_i$ ($i \in [1, N]$) belong to $L^\infty(\Omega)$, and the two boundary conditions $u|_{\Gamma_0} = 0$ and $(\partial u/\partial n)|_{\Gamma_0} = 0$, then $u = 0$ in $\Omega$. In lemma 1, we are in the simple case $a = 0$, $b_i = 0$.

A regularity argument will enable one to derive the same result for $u \in H^1(\Omega)$ satisfying the same equation (with $a = 0$, $b_i = 0$) and the same boundary conditions. We consider the set $\Omega_\rho = \Omega/\{x \in \Omega\,;\, d(x,\Gamma_1) \le \rho\}$ and a cut-off function $\phi \in C^2_0(\mathbb{R}^N)$ which equals 1 on $\overline{\Omega_\rho}$ and vanishes in a (volumic) vicinity of $\overline{\Gamma_1}$. Let $\tilde{u}$ be the extension by 0 of $u$ outside $\Omega$.

By applying the first and second Green formulas, it is easy to see that the function $\phi\tilde{u}$ belongs to $H^1(\Delta,\mathbb{R}^N) = H^2(\mathbb{R}^N)$. Since $\phi = 1$ on $\overline{\Omega_\rho}$, then $u \in H^2(\Omega_\rho)$, $\forall \rho > 0$. Using now the theorem described above, we deduce that $u = 0$ in $\Omega_\rho$, $\forall\rho > 0$. Using Lebesgue's theorem, it follows that $u = 0$ in $\Omega$. ■

## 3. A classical quasi-reversibility formulation

The quasi-reversibility method is due to Lions and Lattès (see [20]). We first present a formulation of quasi-reversibility which is slightly different from the ones presented in [20] (see in particular the formulation proposed in section 8.4, which is the third presented by the authors, the difference between their formulation and the one described here consisting in the variational form).

We first suppose that $(g_0, g_1) \in \mathcal{H} \subset H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$, $\mathcal{H}$ being the set of pairs $(g_0, g_1)$ for which there exists $u_0 \in H^1(\Delta,\Omega)$ such that

$$\begin{cases} u_0|_{\Gamma_0} = g_0 \\ \dfrac{\partial u_0}{\partial n}|_{\Gamma_0} = g_1. \end{cases} \tag{3}$$

Setting

$$H^1(\Delta,\Omega,\Gamma_0) = \{u \in H^1(\Delta,\Omega)\,;\, u|_{\Gamma_0} = 0,\ \frac{\partial u}{\partial n}|_{\Gamma_0} = 0\},$$

and

$$\tilde{H}^1(\Delta,\Omega,\Gamma_0) = \{u \in H^1(\Delta,\Omega)\,;\, u|_{\Gamma_0} = g_0,\ \frac{\partial u}{\partial n}|_{\Gamma_0} = g_1\},$$

the quasi-reversibility method consists of finding an approximation $u_\varepsilon$ of $u$ as a solution of the following weak formulation for small $\varepsilon > 0$ :

Find $u \in \tilde{H}^1(\Delta, \Omega, \Gamma_0)$ such that

$$\int_\Omega \Delta u \Delta v \, dx + \varepsilon \int_\Omega \nabla u . \nabla v \, dx + \varepsilon \int_\Omega uv \, dx = 0, \ \ \forall v \in H^1(\Delta, \Omega, \Gamma_0). \quad (4)$$

***Theorem 1*** *:*

*For a given pair* $(g_0, g_1) \in \mathcal{H}$*, the problem (4) admits a unique solution* $u_\varepsilon$ *in* $\tilde{H}^1(\Delta, \Omega, \Gamma_0)$*. Moreover,*

$$||u_\varepsilon||_{H^1(\Omega)} \leq (1 + \frac{1}{\sqrt{\varepsilon}})||u_0||_{H^1(\Delta,\Omega)}$$

*and*

$$||\Delta u_\varepsilon||_{L^2(\Omega)} \leq 2 \, ||u_0||_{H^1(\Delta,\Omega)}.$$

The proof of theorem 1, which relies on the Lax-Milgram theorem, is standard and will be therefore omitted.

***Theorem 2*** *:*

*For a given pair* $(g_0, g_1) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$ *such that (1) and (2) are satisfied with* $u \in H^1(\Omega)$*, then the solution* $u_\varepsilon$ *of the problem (4) converges to* $u$ *in* $H^1(\Delta, \Omega)$ *as* $\varepsilon$ *tends to* 0*.*

***Proof*** :

From (1) and (4), we obtain

$$(\Delta(u_\varepsilon - u), \Delta v)_{L^2(\Omega)} + \varepsilon(u_\varepsilon, v)_{H^1(\Omega)} = 0, \ \ \forall v \in H^1(\Delta, \Omega, \Gamma_0).$$

Choosing $v = u_\varepsilon - u \in H^1(\Delta, \Omega, \Gamma_0)$ in the previous equation, it follows

$$||(\Delta(u_\varepsilon - u)||^2_{L^2(\Omega)} + \varepsilon(u_\varepsilon, u_\varepsilon - u)_{H^1(\Omega)} = 0, \quad (5)$$

which implies

$$(u_\varepsilon, u_\varepsilon - u)_{H^1(\Omega)} \leq 0$$

and therefore,

$$||u_\varepsilon||_{H^1(\Omega)} \leq ||u||_{H^1(\Omega)}.$$

$H^1(\Omega)$ being separable, there exists a subsequence which is still denoted $u_\varepsilon$ and that weakly converges to a $w$ belonging to $H^1(\Omega)$.

From (5), we also have

$$||(\Delta(u_\varepsilon - u)||^2_{L^2(\Omega)} \leq 2\varepsilon ||u||^2_{H^1(\Omega)}$$

and hence $\Delta u_\varepsilon$ strongly converges to $\Delta u$ in $L^2(\Omega)$.

Since $u_\varepsilon$ weakly converges to $w$ in $H^1(\Omega)$, $u_\varepsilon$ converges to $w$ in $\mathcal{D}'(\Omega)$ (i.e. in the sense of distributions), and hence $\Delta u_\varepsilon$ converges to $\Delta w$ in $\mathcal{D}'(\Omega)$. We thus have $\Delta w = \Delta u = 0$ in $L^2(\Omega)$. Hence, $u_\varepsilon$ weakly converges to $w$ in $H^1(\Delta, \Omega)$.

Besides, the set $\tilde{H}^1(\Delta, \Omega, \Gamma_0)$ is a closed and convex set in the Hilbert space $H^1(\Delta, \Omega)$. Therefore, it is weakly closed (see for example [4]) and we conclude that $w$ satisfies (2).

Lemma 1 then implies $w = u$ and we easily conclude that all the sequence $u_\varepsilon$ weakly converges to $u$ in $H^1(\Omega)$. Now,

$$||u_\varepsilon - u||^2_{H^1(\Omega)} = (u_\varepsilon, u_\varepsilon - u)_{H^1(\Omega)} - (u, u_\varepsilon - u)_{H^1(\Omega)} \leq -(u, u_\varepsilon - u)_{H^1(\Omega)}.$$

Hence $u_\varepsilon$ strongly converges to $u$ in $H^1(\Omega)$ and finally, $u_\varepsilon$ strongly converges to $u$ in $H^1(\Delta, \Omega)$ when $\varepsilon$ tends to 0. ∎

The main drawback of formulation (4) is that any finite element approximation of $u_\varepsilon$ needs to be computed in a finite dimensional subspace of $H^1(\Delta, \Omega)$, and hence needs the use of finite elements of class $C^1$. The following section presents a mixed formulation of quasi-reversibility for which the use of usual finite elements of class $C^0$ is possible.

## 4. A mixed formulation of quasi-reversibility

In this section, we propose a mixed formulation in the sense that the initial fourth-order equation (4) involving a function $u$ in a space of type $H^1(\Delta, \Omega)$ is replaced by a system of two second-order equations involving two functions $u$ and $\lambda$ both in spaces of type $H^1(\Omega)$, similarly as proposed for example in [5] to solve the biharmonic problem. However, this transformation needs the introduction of a second regularization parameter denoted $\delta$, in addition to the classical parameter $\varepsilon$. The necessity of parameter $\delta$ is the point of remark 3.

The main results of that section consist of theorems 3, 4, 5 and 6. The theorem 3 states the well-posedness of the mixed formulation in the case of exact or noisy data $(g_0, g_1)$. In particular, it gives estimates showing stability of the solution with respect to the data. In theorem 4, we establish that if $\varepsilon$ and $\delta$ are linked through a compatibility condition, and if $(g_0, g_1)$ are the uncontaminated data which correspond to the exact solution, then the solution of the mixed formulation converges to that exact solution when $\varepsilon$ tends to 0. In theorem 5, we point out the relationship between the solution of the mixed formulation and the solution of the reference formulation (presented in the previous section), showing that the former one converges to the latter one when $\delta$ tends to 0 for fixed $\varepsilon$. Finally, we consider noisy data in theorem 6, showing that if the discrepancy between these data and the exact ones is bounded by $\sigma$, and if $\sigma$ decreases to 0 as $\varepsilon$, then the solution of the mixed formulation converges to the exact solution when $\varepsilon$ tends to 0.

We define the following spaces $V_0$, $V_1$, $\tilde{V}_0$ :

$$V_0 = \{v \in H^1(\Omega)\,;\, v|_{\Gamma_0} = 0\}, \qquad V_1 = \{v \in H^1(\Omega)\,;\, v|_{\Gamma_1} = 0\},$$

$$\tilde{V}_0 = \{v \in H^1(\Omega)\,;\, v|_{\Gamma_0} = g_0\}.$$

Since $g_0 \in H^{\frac{1}{2}}(\Gamma_0)$, we can define a linear continuous operator $R\ : g_0 \in H^{\frac{1}{2}}(\Gamma_0) \rightarrow u_0 \in H^1(\Omega)$ such that $u_0|_{\Gamma_0} = g_0$ is satisfied (see [15]). Thus there exists a constant $r$ such that

$$||u_0||_{H^1(\Omega)} \leq r\, ||g_0||_{H^{\frac{1}{2}}(\Gamma_0)}. \tag{6}$$

We consider now the following weak formulation for small $\varepsilon > 0$ and for $\delta > 0$ :

Find $(u, \lambda) \in \tilde{V}_0 \times V_1$ such that

$$\left\{ \begin{array}{c} \varepsilon \int_\Omega \nabla u.\nabla v \, dx + \varepsilon \int_\Omega uv \, dx + \int_\Omega \nabla v.\nabla \lambda \, dx = 0, \;\; \forall v \in V_0 \\ \int_\Omega \nabla u.\nabla \mu \, dx - \delta \int_\Omega \nabla \lambda.\nabla \mu \, dx - (1+\delta) \int_\Omega \lambda \mu \, dx = \int_{\Gamma_0} g_1 \mu \, d\Gamma, \;\; \forall \mu \in V_1. \end{array} \right. \tag{7}$$

In (7), the integral on $\Gamma_0$ is defined in the sense of duality pairing between $H^{-\frac{1}{2}}(\Gamma_0)$ and $H^{\frac{1}{2}}_{00}(\Gamma_0)$.

In the sequel, $(\varepsilon, \delta)$ will be denoted $\alpha$.

***Theorem 3*** *:*

*For a given pair $(g_0, g_1) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$, the problem (7) admits a unique solution $(u_\alpha, \lambda_\alpha)$ in $\tilde{V}_0 \times V_1$, with the following estimates :*

$$||u_\alpha||_{H^1(\Omega)} \leq (2 + \frac{1}{\sqrt{\varepsilon\delta}})r||g_0||_{H^{\frac{1}{2}}(\Gamma_0)} + \frac{1}{\sqrt{\varepsilon\delta}}||g_1||_{H^{-\frac{1}{2}}(\Gamma_0)}$$

*and*

$$||\lambda_\alpha||_{H^1(\Omega)} \leq (\sqrt{\frac{\varepsilon}{\delta}} + \frac{1}{\delta})r||g_0||_{H^{\frac{1}{2}}(\Gamma_0)} + \frac{1}{\delta}||g_1||_{H^{-\frac{1}{2}}(\Gamma_0)},$$

*where $r$ is the constant of (6).*

***Proof*** :

Setting $\hat{u} = u - u_0$, the formulation (7) is equivalent to the following one : find $(\hat{u}, \lambda) \in V_0 \times V_1$ such that

$$\left\{ \begin{array}{c} a(\hat{u}, v) + b(v, \lambda) = -a(u_0, v), \;\; \forall v \in V_0 \\ b(\hat{u}, \mu) - c(\lambda, \mu) = l(\mu) - b(u_0, \mu), \;\; \forall \mu \in V_1, \end{array} \right. \tag{8}$$

where

$$\left\{ \begin{array}{ll} a(u, v) = \varepsilon \int_\Omega \nabla u.\nabla v \, dx + \varepsilon \int_\Omega uv \, dx & b(v, \mu) = \int_\Omega \nabla v.\nabla \mu \, dx \\ c(\lambda, \mu) = \delta \int_\Omega \nabla \lambda.\nabla \mu \, dx + (1+\delta) \int_\Omega \lambda \mu \, dx & l(\mu) = \int_{\Gamma_0} g_1 \mu \, d\Gamma. \end{array} \right. \tag{9}$$

$a$ is a continuous bilinear symmetric form on $V_0 \times V_0$, $c$ is a continuous bilinear symmetric form on $V_1 \times V_1$, $b$ is a continuous bilinear form on $V_0 \times V_1$, and $l$ is a continuous linear form on $V_1$.

As $a$ and $c$ are coercive, we can apply the Lax-Milgram theorem to the following problem, which is equivalent to problem (8) : find $(\hat{u}, \lambda) \in V_0 \times V_1$ such that

$$\mathcal{A}((\hat{u}, \lambda); (v, \mu)) = \mathcal{L}(v, \mu), \;\; \forall (v, \mu) \in V_0 \times V_1 \tag{10}$$

where the bilinear form $\mathcal{A}$ is given by

$$\mathcal{A}((\hat{u}, \lambda); (v, \mu)) = a(\hat{u}, v) + b(v, \lambda) - b(\hat{u}, \mu) + c(\lambda, \mu)$$

and the linear form $\mathcal{L}$ by

$$\mathcal{L}(v, \mu) = -a(u_0, v) - l(\mu) + b(u_0, \mu),$$

which ends the proof of existence and uniqueness of $(\hat{u},\lambda)\in V_0\times V_1$ satisfying (8), and then existence and uniqueness of $(u,\lambda)\in\tilde{V}_0\times V_1$ satisfying (7), $(u,\lambda)$ being independent on the choice of $R$.

Setting $(v,\mu)=(\hat{u},\lambda)$ in (10), we obtain

$$a(\hat{u},\hat{u})+c(\lambda,\lambda)=-a(u_0,\hat{u})-l(\lambda)+b(u_0,\lambda),$$

i.e.,

$$\varepsilon||\hat{u}||^2_{H^1(\Omega)}+\delta||\lambda||^2_{H^1(\Omega)}+||\lambda||^2_{L^2(\Omega)}=-\varepsilon(u_0,\hat{u})_{H^1(\Omega)}-\int_{\Gamma_0}g_1\lambda\,d\Gamma+\int_\Omega\nabla u_0.\nabla\lambda\,dx.$$

Then

$$\begin{aligned}\varepsilon||\hat{u}||^2_{H^1(\Omega)}+\delta||\lambda||^2_{H^1(\Omega)}&\le\varepsilon||u_0||_{H^1(\Omega)}||\hat{u}||_{H^1(\Omega)}+||g_1||_{H^{-\frac12}(\Gamma_0)}||\lambda|_{\Gamma_0}||_{H^{\frac12}_{00}(\Gamma_0)}+||u_0||_{H^1(\Omega)}||\lambda||_{H^1(\Omega)}\\&\le\varepsilon\,||u_0||_{H^1(\Omega)}||\hat{u}||_{H^1(\Omega)}+(||u_0||_{H^1(\Omega)}+||g_1||_{H^{-\frac12}(\Gamma_0)})||\lambda||_{H^1(\Omega)}\\&\le\left(\varepsilon\,||u_0||^2_{H^1(\Omega)}+\frac{(||u_0||_{H^1(\Omega)}+||g_1||_{H^{-\frac12}(\Gamma_0)})^2}{\delta}\right)^{\frac12}\left(\varepsilon||\hat{u}||^2_{H^1(\Omega)}+\delta||\lambda||^2_{H^1(\Omega)}\right)^{\frac12}\end{aligned}$$

$$(\varepsilon||\hat{u}||^2_{H^1(\Omega)}+\delta||\lambda||^2_{H^1(\Omega)})^{\frac12}\le\sqrt{\varepsilon}||u_0||_{H^1(\Omega)}+\frac{||u_0||_{H^1(\Omega)}+||g_1||_{H^{-\frac12}(\Gamma_0)}}{\sqrt{\delta}}. \tag{11}$$

From (11), we obtain

$$||\hat{u}||_{H^1(\Omega)}\le(1+\frac{1}{\sqrt{\varepsilon\delta}})||u_0||_{H^1(\Omega)}+\frac{1}{\sqrt{\varepsilon\delta}}||g_1||_{H^{-\frac12}(\Gamma_0)},$$

and using

$$||u||_{H^1(\Omega)}\le||u_0||_{H^1(\Omega)}+||\hat{u}||_{H^1(\Omega)},$$

we obtain the first estimate of theorem 3. The second one is also an obvious consequence of (11). ■

We now analyze convergence of the solution of the mixed formulation of quasi-reversibility to the exact solution in the case of uncontaminated data.

***Theorem 4*** :

*For a given pair* $(g_0,g_1)\in H^{\frac12}(\Gamma_0)\times H^{-\frac12}(\Gamma_0)$ *such that there exists* $u\in H^1(\Omega)$ *satisfying (1) and (2), and if* $\delta$ *is now a bounded function of* $\varepsilon$ *such that*

$$\lim_{\varepsilon\to0}\frac{\varepsilon}{\delta(\varepsilon)}=0, \tag{12}$$

*then the solution* $(u_{\alpha(\varepsilon)},\lambda_{\alpha(\varepsilon)})$ *of the problem (7) converges to* $(u,0)$ *in* $H^1(\Omega)\times H^1(\Omega)$ *as* $\varepsilon$ *tends to* 0.

***Proof*** :

We set $(u_{\alpha(\varepsilon)},\lambda_{\alpha(\varepsilon)})=(u_\varepsilon,\lambda_\varepsilon)$ for sake of simplicity. We remark that $u\in H^1(\Omega)$ satisfies (1) and (2) if and only if $u\in\tilde{V}_0$ and

$$\int_\Omega\nabla u.\nabla\mu\,dx=\int_{\Gamma_0}g_1\mu\,d\Gamma,\ \ \forall\mu\in V_1. \tag{13}$$

Subtracting (13) to the second equation of (7), we obtain

$$\left\{\begin{array}{c}\varepsilon(u_\varepsilon,v)_{H^1(\Omega)}+\int_\Omega\nabla v.\nabla\lambda_\varepsilon\,dx=0,\ \ \forall v\in V_0\\ \int_\Omega\nabla(u_\varepsilon-u).\nabla\mu\,dx-\delta(\varepsilon)(\lambda_\varepsilon,\mu)_{H^1(\Omega)}-(\lambda_\varepsilon,\mu)_{L^2(\Omega)}=0,\ \ \forall\mu\in V_1.\end{array}\right. \tag{14}$$

By setting $v = u_\varepsilon - u \in V_0$ and $\mu = \lambda_\varepsilon \in V_1$ in (14), and by subtracting to each other the two obtained equations, it follows that

$$\varepsilon(u_\varepsilon, u_\varepsilon - u)_{H^1(\Omega)} + \delta(\varepsilon)||\lambda_\varepsilon||^2_{H^1(\Omega)} + ||\lambda_\varepsilon||^2_{L^2(\Omega)} = 0. \tag{15}$$

As in the proof of theorem 2, we conclude from (15) that there exists a subsequence still denoted $u_\varepsilon$ that weakly tends to a $w \in \tilde{V}_0$. We also deduce from (15) that

$$\delta(\varepsilon)\,||\lambda_\varepsilon||^2_{H^1(\Omega)} \leq 2\varepsilon\,||u||^2_{H^1(\Omega)},$$

which implies that $\lambda_\varepsilon$ tends to 0 in $V_1$ because of assumption (12).

Using the second equation of (7), we obtain that

$$\int_\Omega \nabla u_\varepsilon.\nabla\mu\,dx - \delta(\varepsilon)(\lambda_\varepsilon, \mu)_{H^1(\Omega)} - (\lambda_\varepsilon, \mu)_{L^2(\Omega)} = \int_{\Gamma_0} g_1\mu\,d\Gamma, \;\; \forall\mu \in V_1,$$

which implies, taking the limit $\varepsilon \to 0$ (the function $\delta$ being bounded),

$$\int_\Omega \nabla w.\nabla\mu\,dx = \int_{\Gamma_0} g_1\mu\,dx, \;\; \forall\mu \in V_1,$$

and hence $w = u$ from (13) and lemma 1.

We finish the proof the same way as in the proof of theorem 2. ■

***Remark 1*** :

The assumptions of theorem 4 do not imply that $\delta(\varepsilon)$ tends to 0 when $\varepsilon$ tends to 0. The particular case where $\delta$ is a constant fulfills these assumptions.

***Remark 2*** :

The mixed formulation (7) would admit slightly different and even simpler derivations. For example, thank's to the Poincaré-Friedrichs inequality, which implies that in $V_0$ and $V_1$ the semi-norm $|.|_{H^1(\Omega)}$ is actually in those spaces a norm which is equivalent to the norm $||.||_{H^1(\Omega)}$, we could have omitted the terms $(u, v)_{L^2(\Omega)}$ and $(\lambda, \mu)_{L^2(\Omega)}$ in the weak formulation (7).

Now, let us exhibit the relationship between the chosen mixed formulation of quasi-reversibility (i.e. problem (7)) and the reference one (i.e. problem (4)). This relationship is illustrated by the following theorem.

***Theorem 5*** *:*

*For a given pair $(g_0, g_1) \in \mathcal{H}$, for fixed $\varepsilon$ the solution $(u_\alpha, \lambda_\alpha)$ of problem (7) tends to $(u_\varepsilon, -\Delta u_\varepsilon)$ in $H^1(\Omega) \times L^2(\Omega)$ when $\delta$ tends to 0, $u_\varepsilon$ being the solution of problem (4).*

***Proof*** :

We first notice that $u_\varepsilon$ is the solution of problem (4) if and only if $(u_\varepsilon, \phi_\varepsilon = \Delta u_\varepsilon)$ is the solution in $\tilde{\mathcal{K}}$ of the following problem :

$$\int_\Omega \phi_\varepsilon\psi\,dx + \varepsilon\int_\Omega \nabla u_\varepsilon.\nabla v\,dx + \varepsilon\int_\Omega u_\varepsilon v\,dx = 0, \;\; \forall(v, \psi) \in \mathcal{K}, \tag{16}$$

where the sets $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are respectively defined by

$$\mathcal{K} = \{(v, \psi) \in H^1(\Delta, \Omega, \Gamma_0) \times L^2(\Omega)\,|\,\Delta v = \psi\}$$

and

$$\tilde{\mathcal{K}} = \{(v,\psi) \in \tilde{H}^1(\Delta,\Omega,\Gamma_0) \times L^2(\Omega)\,|\,\Delta v = \psi\}.$$

It is easy to see that equivalent definitions of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are

$$\mathcal{K} = \{(v,\psi) \in V_0(\Omega) \times L^2(\Omega)\,|\,\int_\Omega \nabla v.\nabla\mu\,dx + \int_\Omega \psi\mu\,dx = 0,\ \ \forall\mu \in V_1\} \tag{17}$$

and

$$\tilde{\mathcal{K}} = \{(v,\psi) \in \tilde{V}_0(\Omega) \times L^2(\Omega)\,|\,\int_\Omega \nabla v.\nabla\mu\,dx + \int_\Omega \psi\mu\,dx = \int_{\Gamma_0} g_1\mu\,d\Gamma,\ \ \forall\mu \in V_1\}. \tag{18}$$

The formulation (7) can be rewritten as follows :

$$\left\{\begin{array}{c} \varepsilon(u_\alpha,v)_{H^1(\Omega)} + \displaystyle\int_\Omega \nabla v.\nabla\lambda_\alpha\,dx = 0,\ \ \forall v \in V_0 \\ \displaystyle\int_\Omega \nabla u_\alpha.\nabla\mu\,dx - (\lambda_\alpha,\mu)_{L^2(\Omega)} - \delta(\lambda_\alpha,\mu)_{H^1(\Omega)} = \int_{\Gamma_0} g_1\mu\,d\Gamma,\ \ \forall\mu \in V_1. \end{array}\right. \tag{19}$$

As $(u_\varepsilon,\phi_\varepsilon)$ belongs to $\tilde{\mathcal{K}}$, and setting $\lambda_\varepsilon = -\phi_\varepsilon = -\Delta u_\varepsilon$, we have

$$\int_\Omega \nabla u_\varepsilon.\nabla\mu\,dx - \int_\Omega \lambda_\varepsilon\mu\,dx = \int_{\Gamma_0} g_1\mu\,d\Gamma,\ \ \forall\mu \in V_1.$$

Subtracting this equation to the second equation of (19), it follows

$$\int_\Omega \nabla(u_\alpha - u_\varepsilon).\nabla\mu\,dx - (\lambda_\alpha - \lambda_\varepsilon,\mu)_{L^2(\Omega)} - \delta(\lambda_\alpha,\mu)_{H^1(\Omega)} = 0,\ \ \forall\mu \in V_1.$$

Setting now $v = u_\alpha - u_\varepsilon \in V_0$ in the first equation of (19) and $\mu = \lambda_\alpha \in V_1$ in the above equation, it follows

$$\left\{\begin{array}{c} \varepsilon(u_\alpha,u_\alpha - u_\varepsilon)_{H^1(\Omega)} + \displaystyle\int_\Omega \nabla(u_\alpha - u_\varepsilon).\nabla\lambda_\alpha\,dx = 0 \\ \displaystyle\int_\Omega \nabla(u_\alpha - u_\varepsilon).\nabla\lambda_\alpha\,dx - (\lambda_\alpha - \lambda_\varepsilon,\lambda_\alpha)_{L^2(\Omega)} - \delta||\lambda_\alpha||^2_{H^1(\Omega)} = 0. \end{array}\right. \tag{20}$$

Subtracting to each other the two equations of (20), we deduce that

$$\varepsilon(u_\alpha,u_\alpha - u_\varepsilon)_{H^1(\Omega)} + (\lambda_\alpha,\lambda_\alpha - \lambda_\varepsilon)_{L^2(\Omega)} + \delta||\lambda_\alpha||^2_{H^1(\Omega)} = 0. \tag{21}$$

As in theorems 2 and 4, (21) implies that for fixed $\varepsilon$ the sequence $(u_\alpha,\lambda_\alpha)$ is bounded in $H^1(\Omega) \times L^2(\Omega)$ and that there exists a subsequence $(u_\alpha,\lambda_\alpha)$ that weakly converges to a pair $(w,\tau)$ belonging to $\tilde{V}_0 \times L^2(\Omega)$ when $\delta$ tends to 0.

Another consequence of (21) is that

$$||\lambda_\alpha||_{H^1(\Omega)} \leq \sqrt{\frac{2}{\delta}}(\varepsilon\,||u_\varepsilon||^2_{H^1(\Omega)} + ||\lambda_\varepsilon||^2_{L^2(\Omega)})^{\frac{1}{2}}. \tag{22}$$

On the other hand, passing to the limit $\delta \to 0$ in the second equation of (19) using (22) implies that

$$\int_\Omega \nabla w.\nabla\mu\,dx - (\tau,\mu)_{L^2(\Omega)} = \int_{\Gamma_0} g_1\mu\,d\Gamma,\ \ \forall\mu \in V_1.$$

We hence have $(w,-\tau) \in \tilde{\mathcal{K}}$.

The first equation of (19) implies

$$(-\lambda_\alpha,\psi)_{L^2(\Omega)} + \varepsilon(u_\alpha,v)_{H^1(\Omega)} + \int_\Omega \nabla v.\nabla\lambda_\alpha\,dx + (\psi,\lambda_\alpha)_{L^2(\Omega)} = 0,\ \ \forall v \in V_0,\ \ \forall\psi \in L^2(\Omega),$$

and hence

$$(-\lambda_\alpha, \psi)_{L^2(\Omega)} + \varepsilon(u_\alpha, v)_{H^1(\Omega)} = 0, \;\; \forall (v, \psi) \in \mathcal{K}.$$

Taking the limit $\delta \to 0$ in the above equation gives

$$(-\tau, \psi)_{L^2(\Omega)} + \varepsilon(w, v)_{H^1(\Omega)} = 0, \;\; \forall (v, \psi) \in \mathcal{K},$$

which means that $(w, -\tau)$ is the solution of problem (16) and thus $(w, -\tau) = (u_\varepsilon, \Delta u_\varepsilon)$. Finally, all the sequence $(u_\alpha, \lambda_\alpha)$ weakly converges to $(u_\varepsilon, -\Delta u_\varepsilon)$ in $H^1(\Omega) \times L^2(\Omega)$ when $\delta$ tends to 0.

We finish the proof the same way as in the end of the proof of theorem 2. ■

***Remark 3*** :

Reading the statement of theorem 5, one could wonder why not setting $\delta = 0$ in the mixed formulation (7) and then could expect that for $\alpha = (\varepsilon, 0)$, $(u_\alpha, \lambda_\alpha) = (u_\varepsilon, -\Delta u_\varepsilon)$. A deeper analysis of formulation (7) shows that setting $\delta > 0$ is necessary to use the classical existence and uniqueness results on mixed formulation given for example in [5].

In the proof of theorem 3, we saw that (7) is equivalent to a formulation on $V_0 \times V_1$ having the form

$$\begin{cases} a(\hat{u}, v) + b(v, \lambda) = f(v), \;\; \forall v \in V_0 \\ b(\hat{u}, \mu) - c(\lambda, \mu) = g(\mu), \;\; \forall \mu \in V_1, \end{cases}$$

where $f$ and $g$ are continuous linear forms respectively on $V_0$ and $V_1$, the continuous bilinear forms $a$, $b$ and $c$ being defined by (9).

Such general formulation has a chance to be well-posed (see chap. II of [5] for detailed assumptions) in two particular cases :

1 - $a$ and $c$ are coercive,

2 - $a$ is coercive, $c$ is only positive semidefinite, and $b$ satisfies the so-called inf-sup condition.

We recall here what the inf-sup condition is. If $V_0'$ and $V_1'$ denote the dual spaces of $V_0$ and $V_1$, we define the linear operator $B \; : V_0 \to V_1'$ and its transpose $B^t \; : V_1 \to V_0'$ such that

$$< Bv, \mu >_{V_1' \times V_1} = < v, B^t \mu >_{V_0 \times V_0'} = b(v, \mu), \;\; \forall v \in V_0, \;\; \forall \mu \in V_1.$$

The bilinear form $b$ satisfies the inf-sup condition if there exists $k_0 > 0$ such that

$$\sup_{\mu \in V_1} \frac{b(v, \mu)}{||\mu||_{V_1}} \geq k_0 \, ( \inf_{v_0 \in \mathrm{Ker}\, B} ||v + v_0||_{V_0}), \;\; \forall v \in V_0,$$

which is equivalent to the statement that $\mathrm{Im}\, B$ is closed in $V_1'$ (see [5]).

Unfortunately, as following lemma shows, the inf-sup condition does not hold, this fact being strongly related to the ill-posed nature of the Cauchy problem for Laplace's equation.

***Lemma 2*** *:*

$\mathrm{Im}\, B$ *is not closed in* $V_1'$.

***Proof*** :

By definition of $\operatorname{Im} B$,

$$\operatorname{Im} B = \{h \in V_1' \,;\, \exists v \in V_0,\;\; h(\mu) = \int_\Omega \nabla v.\nabla \mu \, dx,\;\; \forall \mu \in V_1\}.$$

The dual space of $L^2(\Omega)$ being identified with itself, the above definition implies that $h \in \operatorname{Im} B \cap L^2(\Omega)$ iff there exists $v \in H^1(\Omega)$ such that

$$-\Delta v = h \;\text{ in }\; \Omega$$

and

$$\left\{ \begin{array}{l} v|_{\Gamma_0} = 0 \\ \dfrac{\partial v}{\partial n}|_{\Gamma_0} = 0. \end{array} \right.$$

In other words, $\operatorname{Im} B \cap L^2(\Omega) = \operatorname{Im} A$, where $A$ is the operator $-\Delta.$ from $H^1(\Delta, \Omega, \Gamma_0)$ to $L^2(\Omega)$.

Due to the ill-posed nature of the Cauchy problem for Laplace's equation, there exists $h \in L^2(\Omega)$ with $h \notin \operatorname{Im} A$. Besides, $\operatorname{Im} A$ is dense in $L^2(\Omega)$ (see for example [20], remark 8.3), which enables one to consider a sequence $h_n$ in $\operatorname{Im} A$ that converges to $h$ in $L^2(\Omega)$. Finally, $h_n$ is a sequence in $\operatorname{Im} B$ which converges in $V_1'$ to an element $h$ which does not belong to $\operatorname{Im} B$, and hence $\operatorname{Im} B$ is not closed in $V_1'$. ■

In conclusion, in the case $\delta = 0$, we satisfy neither the assumptions of case 1 ($c$ is not coercive), nor the ones of case 2 ($c$ is positive semidefinite but $b$ does not satisfy the inf-sup condition). Therefore, the term $-\delta(\lambda, \mu)_{H^1(\Omega)}$ in the formulation (7) can be seen as a regularization term which enables one to restore a well-posed formulation because the assumptions of case 1 are satisfied if $\delta > 0$, that technique being well known as the "penalty method" (see [5]).

We complete this section by analyzing convergence of the solution of the mixed formulation of quasi-reversibility to the exact solution, now in the case of noisy data instead of uncontaminated data. We have the following theorem :

***Theorem 6*** *:*

*Let there be given a pair of uncontaminated data $(g_0, g_1) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$ such that there exists $u \in H^1(\Omega)$ satisfying (1) and (2).*

*Let be given a pair of noisy data $(g_0^\sigma, g_1^\sigma) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$, with*

$$||g_0^\sigma - g_0||_{H^{\frac{1}{2}}(\Gamma_0)} \leq \sigma, \qquad ||g_1^\sigma - g_1||_{H^{-\frac{1}{2}}(\Gamma_0)} \leq c\,\sigma. \tag{23}$$

*Let $(u_\alpha^\sigma, \lambda_\alpha^\sigma)$ be the solution of problem (7), in which $g_0$ and $g_1$ are respectively replaced by $g_0^\sigma$ and $g_1^\sigma$ ($\tilde{V}_0$ is replaced by $\tilde{V}_0^\sigma$).*

*If we finally suppose that $\delta$ is a bounded function of $\varepsilon$ which satisfies the condition (12), that $\sigma$ is also a function of $\varepsilon$ which satisfies $\sigma(\varepsilon)/C\varepsilon \to 1$ when $\varepsilon$ tends to $0$, where $C > 0$ is a constant, then*

$$\lim_{\varepsilon \to 0} ||u_{\alpha(\varepsilon)}^{\sigma(\varepsilon)} - u||_{H^1(\Omega)} = 0,\;\; \lim_{\varepsilon \to 0} ||\lambda_{\alpha(\varepsilon)}^{\sigma(\varepsilon)}||_{H^1(\Omega)} = 0.$$

Theorem 6 shows that it is possible to choose $\alpha = (\varepsilon, \delta)$ as a function of the amplitude of the noise $\sigma$ that makes the solution of the mixed formulation of quasi-reversibility converge to the exact solution for the $H^1(\Omega)$ norm when $\sigma$ tends to 0.

***Proof*** :

Using the estimates of theorem 3 and the estimates (23), we obtain, $(u_\alpha, \lambda_\alpha)$ being the solution of problem (7),

$$||u_\alpha^\sigma - u_\alpha||_{H^1(\Omega)} \leq (2 + \frac{1}{\sqrt{\varepsilon\delta}}) r\,\sigma + \frac{1}{\sqrt{\varepsilon\delta}}\,c\,\sigma$$

and

$$||\lambda_\alpha^\sigma - \lambda_\alpha||_{H^1(\Omega)} \leq (\sqrt{\frac{\varepsilon}{\delta}} + \frac{1}{\delta}) r\,\sigma + \frac{1}{\delta} c\,\sigma.$$

Taking into account the fact that $\delta$ and $\sigma$ are functions of $\varepsilon$ such that $\varepsilon/\delta(\varepsilon) \to 0$ and $\sigma(\varepsilon)/C\varepsilon \to 1$ when $\varepsilon$ tends to 0, we conclude that $u_{\alpha(\varepsilon)}^{\sigma(\varepsilon)} - u_{\alpha(\varepsilon)}$ and $\lambda_{\alpha(\varepsilon)}^{\sigma(\varepsilon)} - \lambda_{\alpha(\varepsilon)}$ tend to 0 in $H^1(\Omega)$ when $\varepsilon$ tends to 0.

We end the proof using theorem 4, which implies that $u_{\alpha(\varepsilon)} - u$ and $\lambda_{\alpha(\varepsilon)}$ tend to 0 in $H^1(\Omega)$ when $\varepsilon$ tends to 0. ■

## 5. A finite element discretization of the mixed formulation

In this section, we derive from the mixed formulation (7) a discretized formulation using classical $C^0$ finite elements based on polynomials of the $k$th degree ($k \geq 1$) and defined on a triangulation of maximum diameter $h$. The main result consists of theorem 7, which shows that if we still consider noisy data such that the discrepancy between these data and the exact ones is bounded by $\sigma$, and if both $\sigma$ and $h^k$ decrease to 0 as $\varepsilon$, then on some regularity assumptions the solution of the discretized mixed formulation converges to the exact solution when $\varepsilon$ tends to 0.

Let be given a finite element space $X_h$ for which the following classical assumptions (see for example [7] for definitions of the different classical notations) are fulfilled :

(i) $\Omega$ is a polygonal ($N = 2$) or polyhedral ($N = 3$) domain
(ii) $\mathcal{T}_h$ is a regular family of triangulations of $\overline{\Omega}$, i.e. satisfying both conditions $h = \max_{K\in\mathcal{T}_h} h_K \to 0$ and $\forall h,\ \max_{K\in\mathcal{T}_h}(h_K/\rho_K) \leq s$ where $s$ is independent on $h$
(iii) $\Gamma_0$ can be written exactly as the union of faces of domains $K \in \mathcal{T}_h$
(iv) the finite element space $X_h$ is such that for each $h$, we associate a family of finite elements $(K, P_K, \Sigma_K)$, $K \in \mathcal{T}_h$, which are affine-equivalent to a single reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ of class $C^0$
(v) the following inclusions are respected : $P_k \subset \hat{P} \subset H^{k+1}(\hat{K})$ for $k \geq 1$.

These assumptions imply that $X_h \subset H^1(\Omega)$. The important point is that the finite element spaces $X_h$ which satisfy assumptions (i) to (v) are classical ($n$-simplices of type $k$ for $k \geq 1$ can do, see [7]), in particular because only a $C^0$ regularity is required through assumption (iv).

We define then

$$X_{0h} = \{v_h \in X_h\,;\, v_h|_{\Gamma_0} = 0\}, \qquad X_{1h} = \{v_h \in X_h\,;\, v_h|_{\Gamma_1} = 0\},$$
$$\tilde{X}_{0h} = \{v_h \in X_h\,;\, v_h - u_{0h} \in X_{0h}\},$$

where $u_{0h}$ is an approximation in $X_h$ of $u_0 = R(g_0)$, and $G_h$ is the space of traces of elements of $X_h$ on $\Gamma_0$. Given these definitions, $X_{0h}$, $X_{1h}$ and $G_h$ are respectively subsets of $V_0$, $V_1$ and $H^{\frac{1}{2}}(\Gamma_0)$. The discrete formulation which corresponds to (7) is the following :

Find $(u_h, \lambda_h) \in \tilde{X}_{0h} \times X_{1h}$ such that

$$\left\{ \begin{array}{c} \varepsilon \int_\Omega \nabla u_h.\nabla v_h \, dx + \varepsilon \int_\Omega u_h v_h \, dx + \int_\Omega \nabla v_h.\nabla \lambda_h \, dx = 0, \ \ \forall v_h \in X_{0h} \\ \int_\Omega \nabla u_h.\nabla \mu_h \, dx - \delta \int_\Omega \nabla \lambda_h.\nabla \mu_h \, dx - (1+\delta) \int_\Omega \lambda_h \mu_h \, dx = \int_{\Gamma_0} g_{1h} \mu_h \, d\Gamma, \ \ \forall \mu_h \in X_{1h}, \end{array} \right. \tag{24}$$

where $g_{1h} \in G_h$ is an approximation of $g_1$.

As for the continuous problem (7), the theorem of Lax-Milgram enables one to prove the existence and uniqueness of a solution $(u_{\alpha h}, \lambda_{\alpha h})$ of problem (24).

***Theorem 7*** *:*

*We assume that conditions* $(i)$ *to* $(v)$ *are fulfilled.*

*Let be given a pair of uncontaminated data* $(g_0, g_1) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$ *such that there exists* $u \in H^1(\Omega)$ *satisfying (1) and (2), and* $u_0 = R(g_0)$.

*Let be given a pair of noisy data* $(g_0^\sigma, g_1^\sigma) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{\frac{1}{2}}(\Gamma_0)$, $u_0^\sigma = R(g_0^\sigma)$ *and* $u_1^\sigma = R(g_1^\sigma)$. *The discrepancy between uncontaminated and noisy data is still given by the estimates (23).*

*Let* $(u_{\alpha h}^\sigma, \lambda_{\alpha h}^\sigma)$ *be the solution of problem (24), in which* $u_{0h}$ *and* $g_{1h}$ *are respectively replaced by* $u_{0h}^\sigma$ *and* $g_{1h}^\sigma$ *(*$\tilde{X}_{0h}$ *is replaced by* $\tilde{X}_{0h}^\sigma$*).*

*We assume that* $u - u_0$, $u_0^\sigma$ *and* $u_1^\sigma$ *all belong to* $H^{k+1}(\Omega)$ *(the integer* $k$ *is the one of assumption* $(v)$*), and that* $u_0^\sigma$ *and* $u_1^\sigma$ *considered as functions of* $\sigma$ *are bounded for the norm* $||.||_{H^{k+1}(\Omega)}$.

*If we finally suppose that* $\delta$ *is a bounded function of* $\varepsilon$ *which satisfies the condition (12), that* $\sigma$ *and* $h$ *are also functions of* $\varepsilon$ *which satisfy* $\sigma(\varepsilon)/C\varepsilon \to 1$ *and* $h^k(\varepsilon)/C'\varepsilon \to 1$ *when* $\varepsilon$ *tends to* 0*, where* $C, C' > 0$ *are constants, then*

$$\lim_{\varepsilon \to 0} ||u_{\alpha(\varepsilon)h(\varepsilon)}^{\sigma(\varepsilon)} - u||_{H^1(\Omega)} = 0, \ \ \lim_{\varepsilon \to 0} ||\lambda_{\alpha(\varepsilon)h(\varepsilon)}^{\sigma(\varepsilon)}||_{H^1(\Omega)} = 0.$$

Theorem 7 shows that on some regularity assumptions, it is possible to choose $\alpha = (\varepsilon, \delta)$ as a function of the amplitude of the noise $\sigma$ and a family of triangulations of maximum diameter $h$ also depending on $\sigma$ that make the discrete solution of the mixed formulation of quasi-reversibility converge to the exact solution for the $H^1(\Omega)$ norm when $\sigma$ tends to 0.

To prove theorem 7, we will need the following lemma, which is for example proved in [13].

***Lemma 3*** *:*

*The subspace of functions* $u$ *in* $C^\infty(\overline{\Omega})$ *that vanish in a (volumic) vicinity of* $\overline{\Gamma_1}$ *is dense in* $V_1$.

***Proof*** :

The pair $(u^\sigma_{\alpha h},\lambda^\sigma_{\alpha h})\in\tilde{X}^\sigma_{0h}\times X_{1h}$ is the solution of problem

$$\left\{\begin{array}{c}\varepsilon(u^\sigma_{\alpha h},v_h)_{H^1(\Omega)}+\displaystyle\int_\Omega \nabla v_h.\nabla\lambda^\sigma_{\alpha h}\,dx=0,\ \ \forall v_h\in X_{0h}\\ \displaystyle\int_\Omega \nabla u^\sigma_{\alpha h}.\nabla\mu_h\,dx-\delta(\lambda^\sigma_{\alpha h},\mu_h)_{H^1(\Omega)}-(\lambda^\sigma_{\alpha h},\mu_h)_{L^2(\Omega)}=\int_{\Gamma_0}g^\sigma_{1h}\mu_h\,d\Gamma,\ \ \forall\mu_h\in X_{1h}.\end{array}\right. \tag{25}$$

The exact solution $u\in\tilde{V}_0$ satisfies

$$\int_\Omega \nabla u.\nabla\mu=\int_{\Gamma_0}g_1\mu\,d\Gamma,\ \ \forall\mu\in V_1. \tag{26}$$

Subtracting equation (26) with $\mu=\mu_h\in V_1$ to the second equation of (25) leads to

$$\int_\Omega \nabla(u^\sigma_{\alpha h}-u).\nabla\mu_h\,dx-\delta(\lambda^\sigma_{\alpha h},\mu_h)_{H^1(\Omega)}-(\lambda^\sigma_{\alpha h},\mu_h)_{L^2(\Omega)}=\int_{\Gamma_0}(g^\sigma_{1h}-g_1)\mu_h\,d\Gamma,\ \ \forall\mu_h\in X_{1h}. \tag{27}$$

If $u^\sigma_{0h}$ is any element of $X_h$, if $w_h$ is any element of $X_{0h}$, setting

$$v_h=u^\sigma_{\alpha h}-u^\sigma_{0h}-w_h=(u^\sigma_{\alpha h}-u)-(u^\sigma_{0h}+w_h-u)\in X_{0h}$$

in the first equation of (25) on the one hand, setting $\mu_h=\lambda^\sigma_{\alpha h}\in X_{1h}$ in equation (27) on the other hand, and subtracting to each other the two obtained equations, gives

$$\begin{aligned}&\varepsilon(u^\sigma_{\alpha h},u^\sigma_{\alpha h}-u)_{H^1(\Omega)}+\delta||\lambda^\sigma_{\alpha h}||^2_{H^1(\Omega)}+||\lambda^\sigma_{\alpha h}||^2_{L^2(\Omega)}\\ &=\varepsilon(u^\sigma_{\alpha h},u^\sigma_{0h}+w_h-u)_{H^1(\Omega)}+\int_\Omega\nabla(u^\sigma_{0h}+w_h-u).\nabla\lambda^\sigma_{\alpha h}\,dx-\int_{\Gamma_0}(g^\sigma_{1h}-g_1)\lambda^\sigma_{\alpha h}\,d\Gamma.\end{aligned}$$

Hence we obtain the following estimate :

$$\begin{aligned}&\varepsilon(u^\sigma_{\alpha h},u^\sigma_{\alpha h}-u)_{H^1(\Omega)}+\delta||\lambda^\sigma_{\alpha h}||^2_{H^1(\Omega)}\\ &\le||u^\sigma_{0h}+w_h-u||_{H^1(\Omega)}(\varepsilon||u^\sigma_{\alpha h}||_{H^1(\Omega)}+||\lambda^\sigma_{\alpha h}||_{H^1(\Omega)})+||g^\sigma_{1h}-g_1||_{H^{-\frac12}(\Gamma_0)}||\lambda^\sigma_{\alpha h}||_{H^1(\Omega)}.\end{aligned} \tag{28}$$

In order to find an estimate of $||u^\sigma_{0h}+w_h-u||_{H^1(\Omega)}$, we write

$$u^\sigma_{0h}+w_h-u=(u^\sigma_{0h}-u^\sigma_0)+(u^\sigma_0-u_0)+(w_h-(u-u_0)).$$

Since $u^\sigma_0\in H^{k+1}(\Omega)$ and $u-u_0\in H^{k+1}(\Omega)\cap V_0$, there exist $u^\sigma_{0h}\in X_h$, $w_h\in X_{0h}$ (see [7] for proof) and constants $c_0$, $c'_0$ independent on $\sigma$ and $h$ such that

$$||u^\sigma_{0h}-u^\sigma_0||_{H^1(\Omega)}\le c_0\,h^k|u^\sigma_0|_{H^{k+1}(\Omega)}$$

$$||w_h-(u-u_0)||_{H^1(\Omega)}\le c'_0\,h^k|u-u_0|_{H^{k+1}(\Omega)}.$$

Taking the first estimate of (23) into account, the continuity of $R$ and the fact that $|u^\sigma_0|_{H^{k+1}(\Omega)}$ is bounded with respect to $\sigma$, we obtain that there exists a constant $C_0$ independent on $\sigma$ and $h$ such that

$$||u^\sigma_{0h}+w_h-u||_{H^1(\Omega)}\le C_0(h^k+\sigma).$$

Considering now $||g^\sigma_{1h}-g_1||_{H^{-\frac12}(\Gamma_0)}$, we write

$$g^\sigma_{1h}-g_1=(g^\sigma_{1h}-g^\sigma_1)+(g^\sigma_1-g_1).$$

Since $u^\sigma_1\in H^{k+1}(\Omega)$, there exists $u^\sigma_{1h}\in X_h$ and a constant $c_1$ independent on $\sigma$ and $h$ such that

$$||u^\sigma_{1h}-u^\sigma_1||_{H^1(\Omega)}\le c_1\,h^k|u^\sigma_1|_{H^{k+1}(\Omega)}.$$

Setting $g_{1h}^\sigma = u_{1h}^\sigma|_{\Gamma_0} \in G_h$, we obtain there exists a constant $c_1'$ independent on $\sigma$ and $h$ such that

$$||g_{1h}^\sigma - g_1^\sigma||_{H^{\frac{1}{2}}(\Gamma_0)} \leq c_1'\, h^k |u_1^\sigma|_{H^{k+1}(\Omega)}.$$

Taking the second estimate of (23) into account and the fact that $|u_1^\sigma|_{H^{k+1}(\Omega)}$ is bounded with respect to $\sigma$, we obtain that there exists a constant $C_1$ independent on $\sigma$ and $h$ such that

$$||g_{1h}^\sigma - g_1||_{H^{-\frac{1}{2}}(\Gamma_0)} \leq C_1(h^k + \sigma).$$

From (28) we obtain that there exists a constant $C_2$ independent on $\sigma$ and $h$ such that

$$\varepsilon(u_{\alpha h}^\sigma, u_{\alpha h}^\sigma - u)_{H^1(\Omega)} + \delta||\lambda_{\alpha h}^\sigma||^2_{H^1(\Omega)} \leq C_2(h^k + \sigma)(\varepsilon||u_{\alpha h}^\sigma||_{H^1(\Omega)} + ||\lambda_{\alpha h}^\sigma||_{H^1(\Omega)}). \tag{29}$$

We introduce now the assumptions that $\delta$ is a bounded function of $\varepsilon$ satisfying $\varepsilon/\delta \to 0$ when $\varepsilon$ tends to 0, that $\sigma$ and $h$ are also functions of $\varepsilon$ satisfying $\sigma(\varepsilon)/C\varepsilon \to 1$ and $h^k(\varepsilon)/C'\varepsilon \to 1$ when $\varepsilon$ tends to 0. These assumptions imply from (29) the existence of another constant $C$ such that

$$\varepsilon||u_{\alpha h}^\sigma||^2_{H^1(\Omega)} + \delta||\lambda_{\alpha h}^\sigma||^2_{H^1(\Omega)} \leq C\varepsilon.$$

We then conclude that $u_{\alpha h}^\sigma$ is bounded in $H^1(\Omega)$ and that $\lambda_{\alpha h}^\sigma$ tends to 0 in $H^1(\Omega)$ when $\varepsilon$ tends to 0.

There exists a subsequence still denoted $u_{\alpha h}^\sigma$ which weakly converges to $w \in H^1(\Omega)$. Passing to the limit $\varepsilon \to 0$ in equation (27) gives

$$\int_\Omega \nabla(w-u).\nabla\mu_h\, dx = 0, \;\; \forall \mu_h \in X_{1h}.$$

We consider $\mu \in H^2(\Omega) \cap V_1$. There exists $\mu_h \in X_{1h}$ and a constant $c$ such that

$$||\mu - \mu_h||_{H^1(\Omega)} \leq c\, h,$$

which in particular implies that $\mu_h$ tends to $\mu$ in $H^1(\Omega)$ when $\varepsilon$ tends to 0. Thus

$$\int_\Omega \nabla(w-u).\nabla\mu\, dx = 0, \;\; \forall \mu \in H^2(\Omega) \cap V_1,$$

and since $H^2(\Omega)\cap V_1$ is dense in $V_1$, which is a consequence of lemma 3, the previous statement holds for any $\mu$ in $V_1$.

Besides, $u_{\alpha h}^\sigma - u_{0h}^\sigma \in X_{0h} \subset V_0$. Since $u_{0h}^\sigma$ strongly converges to $u_0$ in $H^1(\Omega)$, the weak limit of $u_{\alpha h}^\sigma - u_{0h}^\sigma$ in $H^1(\Omega)$ is $w - u_0 \in V_0$. From lemma 1, we conclude that $w = u$ and therefore that all the sequence $u_{\alpha h}^\sigma$ weakly converges to $u$ in $H^1(\Omega)$.

From (29), we deduce that

$$||u_{\alpha h}^\sigma - u||^2_{H^1(\Omega)} \leq |(u, u_{\alpha h}^\sigma - u)_{H^1(\Omega)}| + C\frac{h^k + \sigma}{\varepsilon}(\varepsilon||u_{\alpha h}^\sigma||_{H^1(\Omega)} + ||\lambda_{\alpha h}^\sigma||_{H^1(\Omega)}),$$

which shows that $u_{\alpha h}^\sigma$ strongly converges to $u$ in $H^1(\Omega)$ and ends the proof. ■

***Remark 4*** :

We have assumed in theorem 7 that $g_0^\sigma$ and $g_1^\sigma$ belong to the same space, as well as their discrete approximations $g_{0h}^\sigma = u_{0h}^\sigma|_{\Gamma_0}$ and $g_{1h}^\sigma$. This restrictive choice was done for sake of simplicity, but less regularity could be required for $g_1^\sigma$ and $g_{1h}^\sigma$.

## 6. A numerical example

We consider for $N = 2$ the square domain $\Omega = ]0,1[\times]0,1[ \subset \mathbb{R}^2$ and the harmonic function $u(x,y) = -yx^2 + y^3/3$ in $\Omega$. $u$ will be called the exact solution in the sequel. We denote $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, $D = (0,1)$, $M(\frac{1}{2},0)$ and $N(\frac{1}{2},1)$ in the $(x,y)$ coordinates.

We consider the 3 following cases :

- case 1 : $\Gamma_0 = [A,B] \cup [B,C] \cup [C,D]$ (3 sides of the boundary out of 4)
- case 2 : $\Gamma_0 = [M,B] \cup [B,C] \cup [C,N]$ (half of the boundary)
- case 3 : $\Gamma_0 = [B,C]$ (1 side of the boundary out of 4).

We compute artificial data $g_0$ and $g_1$ on $\Gamma_0$ from the exact solution $u$ and we perform the mixed formulation of quasi-reversibility described in the two last sections.

The numerical results presented in the sequel are obtained using the Finite Element code Melina (see [22]). First of all, no noise contaminating $g_0$ and $g_1$ is introduced and the results are obtained using finite elements based on $P_2$ polynomials, on a $20\times 20$ mesh ($h^k = 1/20^2 = 2.5 \times 10^{-3}$), and with $\varepsilon = \delta = 10^{-4}$. The obtained quasi-reversibility solution (Q.R. solution) $u_{\alpha h}$ is compared to the exact solution through the ratio

$$err = \frac{||u - u_{\alpha h}||_{H^1(\Omega)}}{||u||_{H^1(\Omega)}}.$$

The figures (1), (2), (3) and (4) represent respectively the exact solution and the Q.R. solutions obtained in the cases 1, 2 and 3 described above, and their legend indicates the error $err$ obtained in each case.

Secondly, noise contaminating $g_0$ is introduced, the noise level being 5% or 10%. As indicates theorem 7, better numerical results are obtained by adjusting $h^k$, $\varepsilon$ and $\delta$ to the amplitude of noise $\sigma$, that's why we use now finite elements based on $P_1$ polynomials on the same mesh ($h^k = 1/20^1 = 0.05$), with $\varepsilon = \delta = 0.05$ when the noise level is 5%, $\varepsilon = \delta = 0.1$ when the noise level is 10%. The figures (5) and (6) represent respectively the Q.R. solutions obtained in the case 1 described above for the two different noise levels.

The results highlight in particular how the quality of Q.R. solutions degenerates when less and less data are available, particularly when the distance to $\Gamma_0$ increases. However, robustness of the Q.R. solution with respect to noise contaminating data is quite good. Results could have been improved by optimizing the choices of $h^k$, $\varepsilon$ and $\delta$ separately in each case.

## 7. Conclusion

The mixed formulation of quasi-reversibility we have presented in this paper provides an efficient tool to solve the Cauchy problem for Laplace's equation in the general framework of finite element methods, and thus is well adapted to domains of complex geometry. Contrary to the classical formulation of quasi-reversibility, it can be

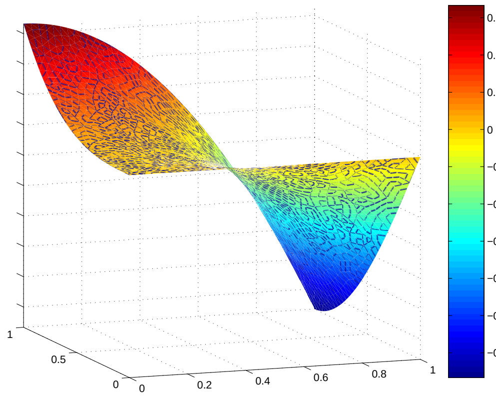


**Figure 1.** *Exact solution*

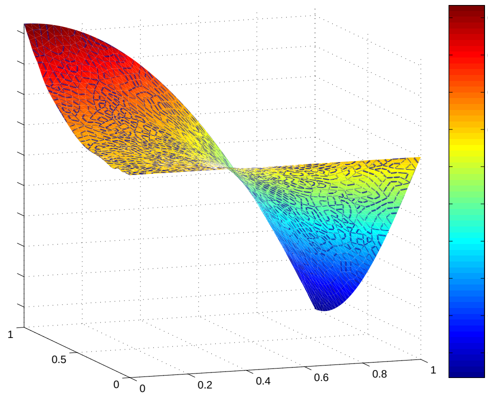


**Figure 2.** *Q.R. solution in the case* $1$ : $err = 0.021$

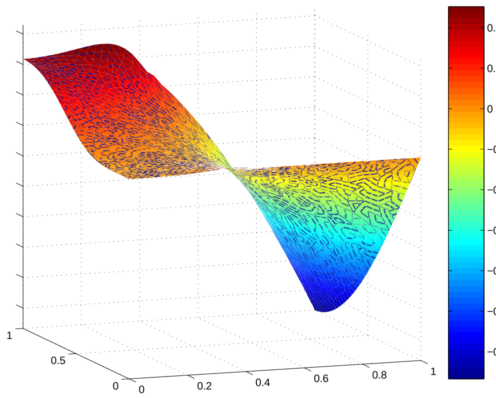


**Figure 3.** *Q.R. solution in the case* $2$ : $err = 0.161$

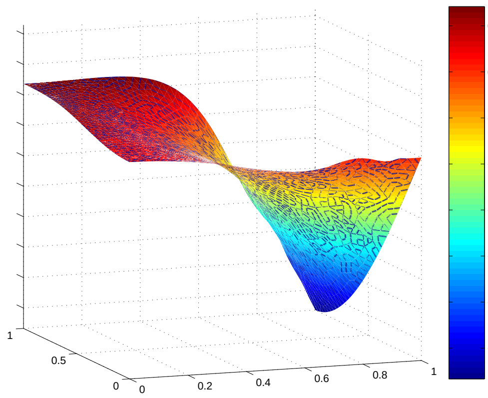


**Figure 4.** *Q.R. solution in the case* $3$ : $err = 0.277$

performed with the usual finite elements of class $C^0$, which are available in almost all Finite Element softwares. It is clear that both classical and mixed formulations can be with no difficulty extended to more general elliptic problems, for example for Helmholtz equation and for the system of elasticity.

Concerning the discretization of the mixed formulation, we indicated how the regularization parameters of the method and the size of the mesh should depend on the amplitude of noise, in order to have convergence of the solution of the discrete mixed formulation to the exact solution when this amplitude tends to 0.

We did not however address the question of the rate of convergence of the mixed formulation to the exact one. A method using Carleman-type estimates like in [18] would probably help to handle that important question. Furthermore, the question of the a

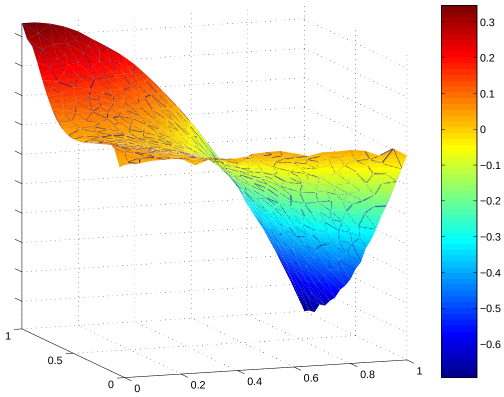


**Figure 5.** *Q.R. solution in the case* 1*,* 5% *noise level :* $err = 0.154$

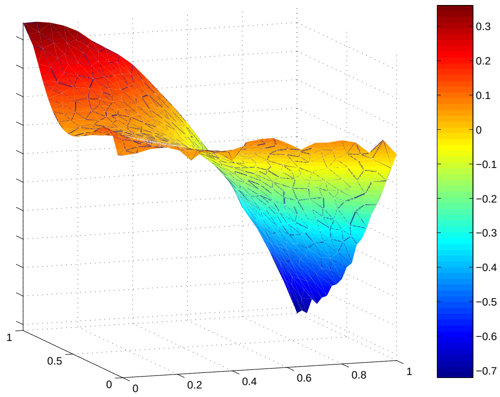


**Figure 6.** *Q.R. solution in the case* 1*,* 10% *noise level :* $err = 0.254$

priori choice of $\varepsilon$, depending for example on the amplitude of noise when it is known, is not addressed in this paper either. We could show that the method of quasi-reversibility can be seen as a Tikhonov regularization of a non-bounded closed operator, i.e. in a more general situation than in the cases of continuous or even compact operators, for which some methods exist to correctly choose the parameter $\varepsilon$ in the Tikhonov regularization. For instance, the so-called discrepancy principle (see [10]) enables to fix that choice in the case of compact operators. The choice of $\varepsilon$ in the method of quasi-reversibility seems to be more difficult than in the continuous or the compact cases, and deserves some further investigations.

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